Optimization of Communication Systems (EE 669)
Assignment 1

March 2022

Problem 1

You are given an affine function $f : \mathbb{R}^n \to \mathbb{R}$, defined as

$$f(x) = a^{\mathrm{T}}x + b,$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Let $f_{\mathtt{FO}}$ denote the first-order approximation of f at an arbitrary $x_0 \in \mathbb{R}^n$. Show that $f(x) = f_{\mathtt{FO}}(x)$ for all $x \in \mathbb{R}^n$.

Problem 2

You are given a quadratic function $f : \mathbb{R}^n \to \mathbb{R}$, defined as

$$f(x) = x^{\mathrm{T}}Px + q^{\mathrm{T}}x + b,$$

where $P \in \mathbb{S}^n$, $q \in \mathbb{R}^n$, and $b \in \mathbb{R}$. Let $f_{\mathtt{SO}}$ denote the second-order approximation of f at an arbitrary $x_0 \in \mathbb{R}^n$. Show that $f(x) = f_{\mathtt{SO}}(x)$ for all $x \in \mathbb{R}^n$.

Problem 3

Show that the nullspace of a matrix $A \in \mathbb{R}^{m\times n}$

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

is a subspace.

Problem 4

Show that the range of a matrix $A \in \mathbb{R}^{m\times n}$

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

is a subspace.

Problem 5

It is often useful to split a vector x into two orthogonal vectors u and v such that $u+v = x$, u is in the direction of a vector w, and v in a direction orthogonal to w. You may assume that all the vectors considered are in $\mathbb{R}^2$. Show that

$$u = \frac{ww^{\mathrm{T}}}{w^{\mathrm{T}}w}\,x \quad \text{and} \quad v = \left(I - \frac{ww^{\mathrm{T}}}{w^{\mathrm{T}}w}\right)\,x,$$

where I is the 2×2 identity matrix. Suppose the matrix $P \in \mathbb{R}^{2\times 2}$ is defined as

$$P = \frac{ww^{\mathrm{T}}}{w^{\mathrm{T}}w}.$$

Show that P is an *idempotent matrix*, i.e., $P^2 = P$. Verify the same for $(I - P)$.

Problem 6

Show that the range of matrix $A \in \mathbb{R}^{m\times n}$

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

is an affine set.

Problem 7

Show that the range of matrix $A \in \mathbb{R}^{m\times n}$

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

is convex.

Problem 8

Show that the set of solution $\mathcal{S}$ of a linear system, i.e.,

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\},\ A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m,$$

is affine.

Problem 9

Show that the set of solution $\mathcal{S}$ of a linear system, i.e.,

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\},\ A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m,$$

is convex.

Problem 10

Consider the problem

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{16}(x_1 - 2)^2 + \dfrac{1}{4}(x_2 - 1)^2 \\ \text{subject to} & x_1 + x_2 - 1 = 0, \end{array} \tag{1}$$

where the decision variable is $x = (x_1, x_2) \in \mathbb{R}^2$. Graphically solve (1) to find the optimal solution $x^\star \in \mathbb{R}^2$ and the optimal value $p^\star$ of the problem.

Problem 11

Let $\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\}$, where $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$. Is $\mathcal{S}$ convex? Justify your answer.

Problem 12

Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = (-\infty, 5]$ and $\mathcal{S}_2 = \{6\}$. Is $\mathcal{S}$ convex? Justify your answer.

Problem 13

Consider the function $f(x) = ||x-b||_2^2$, where $b \in \mathbb{R}^m$. Show that f is convex by restricting it to a line. Recall that $||\cdot||_2$ denotes that ℓ_2-norm and $||a||_2^2 = a^{\mathrm{T}}a$.

Problem 14

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{array} \tag{2}$$

1. Make a sketch of the feasible set.
2. Is problem (2) convex? Justify your answer.
3. Determine the optimal solution of problem (2) by using the sketch in part (a) above.
4. Determine the optimal value of problem (2).

Problem 15

Let $f_0(x_1, x_2) = x_1^2 + x_2^2$ and $f_1(x_1, x_2) = -(x_1 + x_2) + 1$. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_1(x_1, x_2) \leq 0. \end{array} \tag{3}$$

by using a sketch, find the optimal solution $x^\star = (x_1^\star, x_2^\star)$ and the optimal value $p^\star$ of problem (3).

Problem 16

Is $\mathcal{S} = \{x \mid a^{\mathrm{T}}x \leq b, c^{\mathrm{T}}x \geq d\}$, where $a, c \in \mathbb{R}^n$ and $b, d \in \mathbb{R}$ convex? Justify your answer.

Problem 17

Let $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$, where $\mathcal{S}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 + 1 \geq 0, x_2 \geq 0\}$ and $\mathcal{S}_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 + 1 \leq 0, x_2 \leq 0\}$. Is $\mathcal{S}$ convex? Justify your answer.

Problem 18

Let $\mathcal{S} = \{(x,t) \in \mathbb{R}^{n+1} \mid ||x||_2 \leq t\}$. Is $\mathcal{S}$ convex? Justify your answer.

Problem 19

Consider the function $f(x) = x^{\mathrm{T}}Px + q^{\mathrm{T}}x + r$, where P is a real $n \times n$ symmetric *positive definite* matrix [i.e., $P \succ 0$], $q \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Show that f is convex by restricting it to a line.

Problem 20

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}}P^{-1}x + q^{\mathrm{T}}x + r \\ \text{subject to} & l \preceq x \preceq u, \end{array} \tag{4}$$

where the decision variable is $x \in \mathbb{R}^2$,

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad r = 1, \quad l = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and the vector inequality $\preceq$ above is to be intended element-wise.

1. Sketch the feasible set.
2. Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (4).

Problem 21

Consider the following convex optimization problem

$$\begin{array}{ll} \text{minimize} & a^{\mathrm{T}}x \\ \text{subject to} & \frac{1}{2}\, x^{\mathrm{T}}Px \leq 5, \end{array} \tag{5}$$

with variable $x \in \mathbb{R}^2$. The real 2×2 matrix P is symmetric and positive definite. Moreover, is given that $a \in \mathbb{R}^2$. If the matrix P and the vector a are given by

$$P = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{10} \\ \sqrt{10} \end{bmatrix},$$

determine the optimal solution $x^\star$ and the optimal value $p^\star$ of problem (5).

Problem 22

Let $\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\}$, where $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$. Is $\mathcal{S}$ convex? Justify your answer.

Problem 23

Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = (-\infty, 5]$ and $\mathcal{S}_2 = \{6\}$. Is $\mathcal{S}$ convex? Justify your answer.

Problem 24

Consider the set $\mathcal{X} \subseteq \mathbb{R}^2$ defined by the following inequalities:

$$[(x_1 \geq x_2 - 1) \text{ and } x_2 \geq 0] \text{ or } [(x_1 \leq x_2 - 1) \text{ and } x_2 \leq 0].$$

(a) Draw the set $\mathcal{X}$.

(b) Is it convex? Justify your answer.

(c) Show that it can be described as a single quadratic inequality of the form $q(x) = x^{\mathrm{T}}Ax + 2b^{\mathrm{T}}x + c \leq 0$, for a matrix $A = A^{\mathrm{T}} \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$ and $c \in \mathbb{R}$ which you will determine.

(d) What is the convex hull of $\mathcal{X}$?

Problem 25

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}}P^{-1}x + q^{\mathrm{T}}x + r \\ \text{subject to} & l \leq x \leq u, \end{array} \tag{6}$$

where the decision variable is $x \in \mathbb{R}^2$,

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, \quad q = \begin{bmatrix} -10 \\ -2/3 \end{bmatrix}, \quad r = 26, \quad l = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

1. Sketch the feasible set.
2. Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (6).

Problem 26

Consider the set of points $\mathcal{V}$ that are closer (in Euclidean norm) to x_0 than x_i, $i = 1, \ldots, m$. That is, for any $x \in \mathcal{V}$, we have

$$||x - x_0||_2 \leq ||x - x_i||_2, \quad i = 1, \ldots, m.$$

Show that $\mathcal{V}$ is a polyhedron by expressing it in the form $\mathcal{V} = \{x \mid Ax \preceq b\}$, for a matrix A and a vector b which you will determine. Note that the vector inequality $\preceq$ is to be intended elementwise.

Problem 27

Show that $f(X) = \mathtt{tr}(X^{-1})$ is convex on $\mathtt{dom}\ f = \mathbf{S}^n_{++}$ by restricting the function to a line.
Hint: trace of the product of matrices is invariant under cyclical permutations, i.e., $\mathtt{tr}(AB) = \mathtt{tr}(BA)$, where the matrices A and B are of appropriate sizes.

Problem 28

Let $\mathcal{L}$ be a line given by

$$\mathcal{L} = \{x_0 + tu \mid t \in \mathbb{R}\},$$

where $x_0, u \in \mathbb{R}^n$. One needs to compute a point on $\mathcal{L}$ that is closest (in ℓ_2-norm) to a given point $y_0 \in \mathbb{R}^n$. More specifically, the projection of a point y_0 onto a set $\mathcal{L}$, denoted by $\Pi_{\mathcal{L}}(y_0)$ is to be computed.

1. Formulate the above requirement as a mathematical optimization problem by carefully identifying the decision variables.

2. Show that

$$\Pi_{\mathcal{L}}(y_0) = x_0 + \frac{1}{u^{\mathrm{T}}u}\, uu^{\mathrm{T}}(y_0 - x_0)$$

Problem 29

Show that *any* $x \in \mathcal{R}(A)$ is *orthogonal* to *any* vector y in $\mathcal{N}(A^{\mathrm{T}})$, where $A \in \mathbb{R}^{m\times n}$. in other words, show that $x^{\mathrm{T}}y = 0$ for all $x \in \mathcal{R}(A)$, $y \in \mathcal{N}(A^{\mathrm{T}})$. [1]

Problem 30

Show that *any* x in $\mathcal{R}(A^{\mathrm{T}})$ is *orthogonal* to *any* vector y in $\mathcal{N}(A)$, where $A \in \mathbb{R}^{m\times n}$. in other words, show that $x^{\mathrm{T}}y = 0$ for all $x \in \mathcal{R}(A^{\mathrm{T}})$, $y \in \mathcal{N}(A)$. [2]

Problem 31

Let A denote a $n \times n$ matrix, i.e., $A \in \mathbb{R}^{m\times n}$.

1. For any $x \in \mathbb{R}^n$, show that

$$x^{\mathrm{T}}Ax = x^{\mathrm{T}}A^{\mathrm{T}}x.$$

2. By using the result above, show that

$$x^{\mathrm{T}}Ax = x^{\mathrm{T}}\left(\frac{A + A^{\mathrm{T}}}{2}\right)x. \tag{7}$$

[1] This exercise shows that the $\mathcal{R}(A)$ and $\mathcal{N}(A^{\mathrm{T}})$ are mutually orthogonal subspaces. We use the notation $\mathcal{R}(A) \perp \mathcal{N}(A^{\mathrm{T}})$ to denote this result.

[2] This exercise shows that the $\mathcal{R}(A^{\mathrm{T}})$ and $\mathcal{N}(A)$ are mutually orthogonal subspaces. We use the notation $\mathcal{R}(A^{\mathrm{T}}) \perp \mathcal{N}(A)$ to denote this result.

3. By using (7), argue why a generic quadratic function can be represented as $x^{\mathrm{T}}Bx$ where B is symmetric, i.e., $B \in \mathbb{S}^n$.

Problem 32

Show that the matrix $B = AA^{\mathrm{T}}$ is positive semidefinite.

Problem 33

Show that the matrix $B = A^{\mathrm{T}}A$ is positive semidefinite.

Problem 34

Let $\mathcal{X} = \{a, b, c, d\}$, where a, b, c, d are vectors in $\mathbb{R}^3$. Determine the affine hull of $\mathcal{S}$ when

1. $a = (1, 1, 1)$, $b = 2a$, $c = -3a$, and $d = (0, 0, 0)$.
2. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (2, 4.5, 0)$, and $d = (0, 0, 0)$.
3. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0.5, 0, 0.5)$, and $d = (1/3, 1/3, 1/3)$.
4. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0.5, 0, 0.5)$, and $d = (0, 0, 0)$.

Problem 35

Let $\mathcal{X} = \{a, b, c, d\}$, where a, b, c, d are vectors in $\mathbb{R}^3$. Determine the conic hull of $\mathcal{S}$ when

1. $a = (1, 1, 1)$, $b = 2a$, $c = -3a$, and $d = (0, 0, 0)$.
2. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$, and $d = (0, 0, 0)$.
3. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$, and $d = (1, 1, 1)$.
4. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$, and $d = (0, -1, 0)$.

Problem 36

Let $\mathcal{X} = \{a, b, c, d\}$, where a, b, c, d are vectors in $\mathbb{R}^3$. Determine the convex hull of $\mathcal{S}$ when

1. $a = (1, 1, 1)$, $b = 2a$, $c = -3a$, and $d = (0, 0, 0)$.
2. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$, and $d = (0, 0, 0)$.
3. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$, and $d = (1, 1, 1)$.
4. $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$, and $d = (0, -1, 0)$.

Problem 37

Consider the problem

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{16}(x_1-2)^2+\dfrac{1}{4}(x_2-1)^2 \\ \text{subject to} & 2x_1+x_2+1=0, \end{array} \tag{8}$$

where the decision variable is $x=(x_1,x_2)\in\mathbb{R}^2$. Graphically solve (8) to find the optimal solution $x^\star\in\mathbb{R}^2$ and the optimal value $p^\star$ of the problem.

Problem 38

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & 2x_1+x_2 \\ \text{subject to} & 2x_1+2x_2\geq 1 \\ & x_1+3x_2\geq 1 \\ & x_1\geq 0,\ -x_2\geq 0. \end{array} \tag{9}$$

1. Make a sketch of the feasible set.
2. Is problem (9) convex? Justify your answer.
3. Determine the optimal solution of problem (9) by using the sketch in part (a) above.
4. Determine the optimal value of problem (9).

Problem 39

Let $f_0(x_1,x_2)=x_1^2+2x_2^2-2x_1$ and $f_1(x_1,x_2)=-(x_1+x_2)+1$. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f_0(x_1,x_2) \\ \text{subject to} & f_1(x_1,x_2)\leq 0. \end{array} \tag{10}$$

by using a sketch, find the optimal solution $x^\star=(x_1^\star,x_2^\star)$ and the optimal value $p^\star$ of problem (10).

Problem 40

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}}P^{-1}x+q^{\mathrm{T}}x+r \\ \text{subject to} & l\preceq x\preceq u, \end{array} \tag{11}$$

where the decision variable is $x\in\mathbb{R}^2$,

$$P=\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix},\quad q=\begin{bmatrix} 1 \\ -1 \end{bmatrix},\quad r=1,\quad l=\begin{bmatrix} -2 \\ 0 \end{bmatrix},\ \text{and}\quad u=\begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

and the vector inequality $\preceq$ above is to be intended element-wise.

1. Sketch the feasible set.
2. Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (11).

Problem 41

Consider the following convex optimization problem

$$\begin{array}{ll} \text{minimize} & a^{\mathrm{T}}x \\ \text{subject to} & \frac{1}{2}\, x^{\mathrm{T}}Px + q^{\mathrm{T}}x \leq 5, \end{array} \tag{12}$$

with variable $x \in \mathbb{R}^2$. The real 2×2 matrix P is symmetric and positive definite. Moreover, is given that $a \in \mathbb{R}^2$. If the matrix P, and the vectors a and q are given by

$$P = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{10} \\ \sqrt{10} \end{bmatrix}, \quad q = \begin{bmatrix} \sqrt{5} \\ \sqrt{10} \end{bmatrix},$$

determine the optimal solution $x^\star$ and the optimal value $p^\star$ of problem (17).

Problem 42

Consider the problem

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{16}(x_1 - 2)^2 + \dfrac{1}{4}(x_2 - 1)^2 + + x_2 \\ \text{subject to} & 2x_1 + 3x_2 + 1 = 0, \end{array} \tag{13}$$

where the decision variable is $x = (x_1, x_2) \in \mathbb{R}^2$. Graphically solve (13) to find the optimal solution $x^\star \in \mathbb{R}^2$ and the optimal value $p^\star$ of the problem.

Problem 43

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & 2x_1 - x_2 \\ \text{subject to} & 2x_1 + 2x_2 \geq 1 \\ & x_1 - 3x_2 \geq 1 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{array} \tag{14}$$

1. Make a sketch of the feasible set.
2. Is problem (14) convex? Justify your answer.
3. Determine the optimal solution of problem (14) by using the sketch in part (a) above.
4. Determine the optimal value of problem (14).

Problem 44

Let $f_0(x_1, x_2) = x_1^2 + 2x_2^2 - 2x_1 + 4x_2$ and $f_1(x_1, x_2) = (x_1 - x_2) + 1$. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_1(x_1, x_2) \leq 0. \end{array} \tag{15}$$

by using a sketch, find the optimal solution $x^\star = (x_1^\star, x_2^\star)$ and the optimal value $p^\star$ of problem (15).

Problem 45

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}} P^{-1} x + q^{\mathrm{T}} x + r \\ \text{subject to} & l \preceq x \preceq u, \end{array} \tag{16}$$

where the decision variable is $x \in \mathbb{R}^2$,

$$P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad r = 1, \quad l = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

and the vector inequality $\preceq$ above is to be intended element-wise.

1. Sketch the feasible set.
2. Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (20).

Problem 46

Consider the following convex optimization problem

$$\begin{array}{ll} \text{minimize} & a^{\mathrm{T}} x \\ \text{subject to} & \frac{1}{2}\, x^{\mathrm{T}} P x + q^{\mathrm{T}} x \leq 5, \end{array} \tag{17}$$

with variable $x \in \mathbb{R}^2$. The real 2×2 matrix P is symmetric and positive definite. Moreover, is given that $a \in \mathbb{R}^2$. If the matrix P, and the vectors a and q are given by

$$P = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{20} \\ \sqrt{10} \end{bmatrix}, \quad q = \begin{bmatrix} \sqrt{5} \\ \sqrt{5} \end{bmatrix},$$

determine the optimal solution $x^\star$ and the optimal value $p^\star$ of problem (17).

Problem 47

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & 2x_1 - x_2 \\ \text{subject to} & 2x_1 + 2x_2 \geq 4 \\ & -x_1 + 3x_2 \geq 1 \\ & x_1 + 4 \geq 0, \; x_2 \geq 0. \end{array} \tag{18}$$

1. Make a sketch of the feasible set.
2. Is problem (18) convex? Justify your answer.
3. Determine the optimal solution of problem (18) by using the sketch in part (a) above.
4. Determine the optimal value of problem (18).

Problem 48

Let $f_0(x_1, x_2) = 4x_1^2 + 2x_2^2 - 2x_1 + 4x_2$ and $f_1(x_1, x_2) = (x_1 - x_2) + 1$. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_1(x_1, x_2) \leq 0. \end{array} \tag{19}$$

by using a sketch, find the optimal solution $x^\star = (x_1^\star, x_2^\star)$ and the optimal value $p^\star$ of problem (19).

Problem 49

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}} P x - q^{\mathrm{T}} x + r \\ \text{subject to} & l \preceq x \preceq u, \end{array} \tag{20}$$

where the decision variable is $x \in \mathbb{R}^2$,

$$P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad r = 1, \quad l = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

and the vector inequality $\preceq$ above is to be intended element-wise.

1. Sketch the feasible set.
2. Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (20).

Problem 50

Consider the following convex optimization problem

$$\begin{array}{ll} \text{maximize} & a^{\mathrm{T}} x \\ \text{subject to} & \frac{1}{2}\, x^{\mathrm{T}} P x + q^{\mathrm{T}} x \leq 6, \end{array} \tag{21}$$

with variable $x \in \mathbb{R}^2$. The real 2×2 matrix P is symmetric and positive definite. Moreover, is given that $a \in \mathbb{R}^2$. If the matrix P, and the vectors a and q are given by

$$P = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{20} \\ \sqrt{10} \end{bmatrix}, \quad q = \begin{bmatrix} \sqrt{5} \\ \sqrt{5} \end{bmatrix},$$

determine the optimal solution $x^\star$ and the optimal value $p^\star$ of problem (21).