Optimization of Communication Systems
EE 669

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Lecture 4: Optimization Problems: A Geometrical Point of View

Recall from Lecture 1

- what is optimization?
- we have decision variables, say $x \in \mathbb{R}^n$
- we have an objective; a function f_0 of x
- there are constraints as well
- constraints are again characterized by using functions of x
  - equality constraints: $g_i(x) = 0,\ i = 1, \ldots, p$
  - inequality constraints $f_i(x) \leq 0,\ i = 1, \ldots, m$

Standard Problem Formulation

- problem can be formulated as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & g_i(x) = 0, \;\; i = 1, \ldots, p \\ & f_i(x) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

- decision variable: x
- objective function: $f_0(x)$
- equality constraint: $g_i(x) = 0, \;\; i = 1, \ldots, p$
- inequality constraint: $f_i(x) \leq 0, \;\; i = 1, \ldots, m$

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$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & g_i(x) = 0, \;\; i = 1, \ldots, p \\ & f_i(x) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

- optimal value: $p^\star$

Feasible Set

- the x should be an element in the domain of all the functions, i.e.,

$$x \in \left(\cap_{i=0}^{m} \ \mathtt{dom}\ f_i \right) \cap \left(\cap_{i=1}^{p} \ \mathtt{dom}\ g_i \right) = \mathcal{D}$$

- consider the set $\mathcal{F}$ of points $x \in \mathcal{D}$ that satisfies the constraints, i.e.,

$$\mathcal{F} = \{x \in \mathcal{D} \mid g_1(x) = 0, g_1(x) = 0, \ldots, g_p(x) = 0, \\ f_1(x) \leq 0, f_2(x) \leq 0, \ldots, f_m(x) \leq 0\}$$

Feasible Set

- the set $\mathcal{F}$ is called the feasible set or the constraint set of the underlying optimization problem
- what is needed is to select $x \in \mathcal{F}$ that achieves the best objective value
- a point x in $\mathcal{F}$ is called a feasible point

A Compact Representation

- the foregoing problem can be formulated as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in \mathcal{F} \end{array}$$

- optimal value[1]: $p^\star = \min\limits_{x \in \mathcal{F}} f_0(x)$

[1] Rigorously, the notation '$\inf$' should be used instead of '$\min$'.

Problems in Maximization Form

- some problems come in the form of maximization, i.e.,

$$\begin{array}{ll} \text{maximize} & g_0(x) \\ \text{subject to} & x \in \mathcal{F} \end{array}$$

- optimal value[2]: $p^\star = \max\limits_{x \in \mathcal{F}} g_0(x)$

- note: *maximization problems* $\rightarrow$ *minimization problems* , why?

$$\max_{x \in \mathcal{F}} g_0(x) = - \min_{x \in \mathcal{F}} \left(-g_0(x)\right)$$

[2] Rigorously, the notation '$\inf$' should be used instead of '$\min$' and '$\sup$' should be used instead of '$\max$'.

Infeasibility of a Problem

- the feasible set may be empty, why? the constraints cannot be satisfied simultaneously
- in this case the problem is said to be *infeasible*
- use the convention that the optimal value

$$p^\star = \begin{cases} +\infty & \text{minimization problem is infeasible} \\ -\infty & \text{maximization problem is infeasible} \end{cases}$$

What is a Solution

- we need the optimal value $p^\star$ of the objective function, *possibly* together with a corresponding minimizer
- the minimizer: a vector $x^\star \in \mathcal{F}$ that achieves $p^\star$, i.e.,
  - a vector $x^\star \in \mathcal{F}$ such that $p^\star = f_0(x^\star)$
- we say that the problem is attained if there is such a vector

Optimal Set $\mathcal{X}^\star$

- the set of feasible points for which the problem is attained, i.e.,

$$\mathcal{X}^\star = \{x \in \mathcal{F} \mid f_0(x) = p^\star\}$$

- a point x is said to be optimal if $x \in \mathcal{X}^\star$
- note: $\mathcal{X}^\star$ can be empty as well, e.g.,
  - consider the following minimization problem:

$$\text{minimize} \quad e^x$$

  - here we have $p^\star = 0$, but the problem is not attained

Feasible Set

- determine the feasible set for
  - exercise 1: $f_0(x) = 2x + 1$
  - exercise 2: $f_0(x) = 2x + 1,\ g_1(x) = x - 1$
  - exercise 3: $f_0(x) = 2x + 1,\ g_1(x) = x - 1,\ f_1(x) = 4x - 8$
  - exercise 4: $f_0(x) = 2x + 1,\ f_1(x) = -(x - 1),\ f_2(x) \leq 2x - 12$
  - exercise 5: $f_0(x) = x^2,\ f_1(x) = x - 1,\ f_2(x) = -2x - 4$
  - exercise 6: $f_0(x) = x^2,\ f_1(x) = \sin(x) - (1/2)$
  - exercise 7: $f_0(x) = \log(x),\ f_1(x) = x + 1$
  - exercise 8: $f_0(x) = \log(x + 1),\ f_1(x) = x - 1$

Standard Problem Formulation

- problem can be formulated as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & g_i(x) = 0,\ i = 1, \ldots, p \\ & f_i(x) \leq 0,\ i = 1, \ldots, m \end{array}$$

- decision variable: x
- objective function: $f_0(x)$
- equality constraint: $g_i(x) = 0,\ i = 1, \ldots, p$
- inequality constraint: $f_i(x) \leq 0,\ i = 1, \ldots, m$

Unconstrained Optimization

- let us first check the case without constraint functions
- exercise 9: decision variable is $x \in \mathbb{R}$

$$\text{minimize} \quad x$$

  - determine the feasible set $\mathcal{F}$
  - compute the gradient $\nabla f_0(x)$
  - draw the graph of f_0 and mark the gradient $\nabla f_0(x)$ on the $\mathtt{dom}\ f_0$
  - mark the direction on the domain that one needs to move in order to minimize the function
  - what is the optimal value $p^\star$ and optimal solution $x^\star$

Unconstrained Optimization

- perform the same for
  - exercise 10: $f_0(x) = 2x + 1$, decision variable is $x \in \mathbb{R}$
  - exercise 11: $f_0(x) = x^2$, decision variable is $x \in \mathbb{R}$
  - exercise 12: $f_0(x) = \log(x - 1)$, decision variable is $x \in \mathbb{R}$
  - exercise 13: $f_0(x) = 3x^2 + x + 1$, decision variable is $x \in \mathbb{R}$

Unconstrained Optimization

- let us consider unconstrained optimization with functions on $\mathbb{R}^2$
- exercise 14: $f_0(x) = x_1 + x_2$, $x \in \mathbb{R}^2$
  - determine the feasible set $\mathcal{F}$
  - compute the gradient $\nabla f_0(x)$
  - draw $\mathcal{F}$, the level sets of f_0 and mark the gradient $\nabla f_0(x)$ on the $\mathbf{dom}\ f_0$
  - what is the optimal value $p^\star$ and optimal solution $x^\star$

Unconstrained Optimization

- perform the same for
  - exercise 15: $f_0(x) = q^T x$, $x \in \mathbb{R}^2$, $q = (2, 1)$
  - exercise 16: $f_0(x) = ||x||^2$, $x \in \mathbb{R}^2$
  - exercise 17: $f_0(x) = x^T x + q^T x + 1$, $x \in \mathbb{R}^2$, $q = (1, 1)$
  - exercise 18: $f_0(x) = x^T P x + q^T x + 1$, $x \in \mathbb{R}^2$, $q = (0, 0)$, and

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Equality Constrained Optimization

- let us check the case together with equality constraint functions on $\mathbb{R}$
- exercise 19: decision variable is $x \in \mathbb{R}$

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 2x + 3 = 0, \end{array}$$

  - determine the feasible set $\mathcal{F}$
  - compute the gradient $\nabla f_0(x)$
  - draw the graph of related functions, e.g., $f_0,\ g_1$
  - what is the optimal value $p^\star$ and optimal solution $x^\star$

Equality Constrained Optimization

- perform the same for the following: here $x \in \mathbb{R}$
  - exercise 20: $f_0(x) = 2x + 1,\ g_1(x) = x - 1$
  - exercise 21: $f_0(x) = 2x + 1,\ g_1(x) = x - 1,\ g_2(x) = 2x - 2$
  - exercise 22: $f_0(x) = x^2,\ g_1(x) = x - 1$
  - exercise 23: $f_0(x) = x^2,\ g_1(x) = \sin(x)$
  - exercise 24: $f_0(x) = \log(x + 1),\ g_1(x) = x^2 - 1$
  - exercise 25: $f_0(x) = 3x^2 + x + 1,\ g_1(x) = x^2 + x - 6$

Equality Constrained Optimization

- let us check the case together with functions on $\mathbb{R}^2$
- exercise 26: $f_0(x) = x_1 + x_2$, $g_1(x) = ||x - (1,1)||^2 - 4$
  - determine the feasible set $\mathcal{F}$
  - compute the gradient $\nabla f_0(x)$
  - draw $\mathcal{F}$, and the level sets of f_0
  - what is the optimal value $p^\star$ and optimal solution $x^\star$

Equality Constrained Optimization

- perform the same for the following:
  - exercise 27: $f_0(x) = q^T x$, $q = (-2, 1)$, $g_1(x) = ||x - (1, 1)||^2 - 4$, $g_2(x) = ||x - (1, 2)||^2 - 1$
  - exercise 28: $f_0(x) = ||x||^2$, $g_1(x) = 2x_1 + x_2 - 1$, $g_2(x) = x_1 + x_2 - 1$
  - exercise 29: $f_0(x) = x^T x + q^T x + 1$, $g_1(x) = x_1 + x_2 - 1$, where $q = (1, 1)$,
  - exercise 30: $f_0(x) = x^T P x + q^T x + 1$, $g_1(x) = x_1 + x_2 - 1$, where $q = (0, 0)$, and

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Constrained Optimization

- recall: in this case equality as well as inequality constraint functions are there, i.e., the standard problem formulation

- exercise 31: solve the following problem graphically

$$\begin{array}{ll} \text{minimize} & x^T P x + q^T x + 1 \\ \text{subject to} & s^T x - (1/2) = 0, \\ & 1^T x - 1 \leq 0, \end{array}$$

where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$