Optimization of Communication Systems
EE 669

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Lecture 3: Vector Algebra (Part II)

Splitting Vectors into Two Components

Split a 2- or a 3-Vector

- it is often useful to split a vector $\mathbf{v}$ into two orthogonal vectors such that
  - one in the direction of a vector $\mathbf{w}$ and the other in a direction orthogonal to $\mathbf{w}$
  - the vector $\mathbf{v}$ is the sum of the vector component along $\mathbf{w}$ and the vector component orthogonal to $\mathbf{w}$
- illustrate the idea in a Cartesian plane

Split a 2- or a 3-Vector

- suppose θ is the angle between $\mathbf{v}$ and $\mathbf{w}$
- then the scalar component of $\mathbf{v}$ in the direction of $\mathbf{w}$ is

$$||\mathbf{v}||_2 \cos\theta = \frac{\mathbf{v}^\mathsf{T}\mathbf{w}}{||\mathbf{w}||_2}$$

- illustrate the idea in a Cartesian plane

Split a 2- or a 3-Vector

- hence the vector component of $\mathbf{v}$ along $\mathbf{w}$ is

$$
\begin{aligned}
\left(||\mathbf{v}||_2 \cos\theta\right)\left(\text{unit vector along } \mathbf{w}\right) &= \left(||\mathbf{v}||_2 \cos\theta\right)\frac{\mathbf{w}}{||\mathbf{w}||_2} \\
&= \left(\frac{\mathbf{v}^\mathsf{T}\mathbf{w}}{||\mathbf{w}||_2}\right)\frac{\mathbf{w}}{||\mathbf{w}||_2} \\
&= \left(\frac{\mathbf{v}^\mathsf{T}\mathbf{w}}{||\mathbf{w}||_2^2}\right)\mathbf{w} \\
&= \mathbf{w}\left(\frac{\mathbf{v}^\mathsf{T}\mathbf{w}}{||\mathbf{w}||_2^2}\right) = \mathbf{w}\left(\frac{\mathbf{w}^\mathsf{T}\mathbf{v}}{||\mathbf{w}||_2^2}\right) \\
&= \left(\frac{\mathbf{w}\mathbf{w}^\mathsf{T}}{\mathbf{w}^\mathsf{T}\mathbf{w}}\right)\mathbf{v}
\end{aligned}
$$

- illustrate the idea in a Cartesian plane

Split a 2- or a 3-Vector

- similarly the vector component of $\mathbf{v}$ orthogonal to $\mathbf{w}$ is

$$\begin{aligned}\mathbf{v} - (\text{vector component of } \mathbf{v} \text{ along } \mathbf{w}) &= \mathbf{v} - \left(\frac{\mathbf{v}^{\mathsf{T}}\mathbf{w}}{||\mathbf{w}||_2^2}\right)\mathbf{w} \\ &= \mathbf{v} - \left(\frac{\mathbf{w}\mathbf{w}^{\mathsf{T}}}{\mathbf{w}^{\mathsf{T}}\mathbf{w}}\right)\mathbf{v}\end{aligned}$$

- illustrate the idea in a Cartesian plane
- check the orthogonality between those two components

Split an n-Vector

- the splitting an n-vector $\mathbf{v}$ into two orthogonal vectors such that
  - one in the direction of an n-vector $\mathbf{w}$ and the other in a direction orthogonal to $\mathbf{w}$
  - the vector $\mathbf{v}$ is the sum of the vector component along $\mathbf{w}$ and the vector component orthogonal to $\mathbf{w}$

  follows in an identical manner
- of course, visual concept is not at our disposal in higher Euclidean space

Split an n-Vector

- the vector component of $\mathbf{v}$ along $\mathbf{w}$ is

$$\left(\frac{\mathbf{v}^{\mathsf{T}}\mathbf{w}}{||\mathbf{w}||_2}\right)\frac{\mathbf{w}}{||\mathbf{w}||_2} = \left(\frac{\mathbf{w}\mathbf{w}^{\mathsf{T}}}{\mathbf{w}^{\mathsf{T}}\mathbf{w}}\right)\mathbf{v}$$

- vector component of $\mathbf{v}$ orthogonal to $\mathbf{w}$

$$\mathbf{v} - \left(\frac{\mathbf{v}^{\mathsf{T}}\mathbf{w}}{||\mathbf{w}||_2}\right)\frac{\mathbf{w}}{||\mathbf{w}||_2} = \mathbf{v} - \left(\frac{\mathbf{w}\mathbf{w}^{\mathsf{T}}}{\mathbf{w}^{\mathsf{T}}\mathbf{w}}\right)\mathbf{v}$$

Split an n-Vector

- if $\mathbf{v} = (1, 4)$ and $\mathbf{w} = (1, 1)$ find the corresponding components
- if $\mathbf{v} = (1, 4, 3, 2)$ and $\mathbf{w} = (1, 0, 0, 0)$ find the corresponding components

Lines

Lines

- let us start with lines in $\mathbb{R}^3$
- the concepts directly applies in $\mathbb{R}^2$ (at our disposal)
- the concepts directly applies in $\mathbb{R}^n$ (not at our disposal)

Lines in $\mathbb{R}^3$

- let $\mathbf{v} = (v_1, v_2, v_3)$ be a nonzero vector in $\mathbb{R}^3$
- identify $t\mathbf{v}$ if t is a scalar, i.e., $t \in \mathbb{R}$
- identify the set [1] $\{t\mathbf{v} \mid t \in \mathbb{R}\}$

[1] See p. 35 for details.

Lines in $\mathbb{R}^3$

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- identify the set [1] $\{t\mathbf{v} \mid t \in \mathbb{R}\}$
  - all points on the line through origin in the direction of $\mathbf{v}$ (sketch)

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- let $\boldsymbol{\xi}$ be a vector in $\mathbb{R}^3$
- identify the set $\{\boldsymbol{\xi} + t\mathbf{v} \mid t \in \mathbb{R}\}$

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  - all points on the line through origin in the direction of $\mathbf{v}$ (sketch)
- let $\boldsymbol{\xi}$ be a vector in $\mathbb{R}^3$
- identify the set $\{\boldsymbol{\xi} + t\mathbf{v} \mid t \in \mathbb{R}\}$
  - all points on the line through $\boldsymbol{\xi}$ in the direction of $\mathbf{v}$ (sketch)

[1] See p. 35 for details.

Parametric Equation of Line in $\mathbb{R}^3$

- the parametric equation of the line through $\boldsymbol{\xi}$ in the direction of $\mathbf{v}$

$$\mathbf{x} = \boldsymbol{\xi} + t\mathbf{v}, \quad (t \in \mathbb{R}) \tag{1}$$

- (1) is also known as the vector equation of the line

- each value of the parameter t yields a unique point on the line and vice versa

Cartesian Equations of Line in $\mathbb{R}^3$

- write the parametric equation (1) in terms of vector components
- what are the corresponding scalar equations ?
- eliminate t assuming $\mathbf{v} = (v_1, v_2, v_3) \neq (0, 0, 0)$:

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$$\frac{x_1 - \xi_1}{v_1} = \frac{x_2 - \xi_2}{v_2} = \frac{x_3 - \xi_3}{v_3} \tag{2}$$

- (2) is known as Cartesian equations of the line

Exercise

- what is the distance between $\mathbf{x}$ and $\boldsymbol{\xi}$?

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  - $||\mathbf{x} - \boldsymbol{\xi}||_2 = |t| \; ||v||_2$
- if $\boldsymbol{\xi} = (6, -2, 0)$ and $\mathbf{v} = (4, 3, -1)$, check whether the point $(2, -5, 1)$ lies on the line

Exercise

- what is the distance between $\mathbf{x}$ and $\boldsymbol{\xi}$?
  - $||\mathbf{x} - \boldsymbol{\xi}||_2 = |t| \; ||v||_2$
- if $\boldsymbol{\xi} = (6, -2, 0)$ and $\mathbf{v} = (4, 3, -1)$, check whether the point $(2, -5, 1)$ lies on the line
- if $\boldsymbol{\xi} = (6, -2, 0)$ and $\mathbf{v} = (0, 0, 1)$, determine the Cartesian equations of the corresponding line

Coplanar Lines and Skew Lines

- lines [*cf.* (1),(2)] that lie in the same plane are called coplanar
  - they either intersect or are parallel
  - parallel $\Longrightarrow$ coplanar ?
- lines [*cf.* (1),(2)] that are not coplanar are called skew

Coplanar Lines and Skew Lines

- suppose the vector equations of two lines are
  - $\mathbf{x} = \boldsymbol{\xi} + t\mathbf{v}, \quad (t \in \mathbb{R})$ and $\mathbf{x} = \boldsymbol{\eta} + r\mathbf{u}, \quad (r \in \mathbb{R})$
- the two lines intersect $\iff \exists t, r \in \mathbb{R},\ \boldsymbol{\xi} + t\mathbf{v} = \boldsymbol{\eta} + r\mathbf{u}$ [2]
- if the two lines intersect, the angle between the two lines is the simply the angle between the vectors $\mathbf{v}$ and $\mathbf{u}$

[2] See page 37 for details of quantifiers.

Exercise

- if $\boldsymbol{\xi} = (1, 0, 1)$, $\boldsymbol{\eta} = (0, 0, 0)$, $\mathbf{v} = (-1, -1, -1)$, and $\mathbf{u} = (2, 0, 1)$, check whether the corresponding lines are coplanar or skew

- try the Exercises listed in § 1.8 of Binmore-2012

Planes

Planes

- let us start with planes in $\mathbb{R}^3$
- the concepts directly applies in $\mathbb{R}^2$ (at our disposal)
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Plane Through Origin

- let $\mathbf{v} \in \mathbb{R}^3$ be a fixed nonzero vector
- consider the equation $\mathbf{v}^\mathsf{T}\mathbf{x} = 0$
- the equation characterizes points that are orthogonal to $\mathbf{v}$
- identify the set of points that satisfy the above equation:

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- explain geometrically the idea
- note: $\mathbf{v}$ is normal to the plane

Plane Through a Point $\boldsymbol{\xi}$

- let $\mathbf{v} \in \mathbb{R}^3$ be a fixed nonzero vector
- $\boldsymbol{\xi} \in \mathbb{R}^3$ be fixed vector
- consider the equation $\mathbf{v}^{\mathsf{T}}(\mathbf{x} - \boldsymbol{\xi}) = 0$
- identify the set of points that satisfy the above equation:

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  - a plane through the point $\boldsymbol{\xi}$
- explain geometrically the idea
- note: $\mathbf{v}$ is normal to the plane

Plane Through a Point $\boldsymbol{\xi}$

- consider the set of points that satisfy $\mathbf{v}^{\mathsf{T}}(\mathbf{x} - \boldsymbol{\xi}) = 0$
- note: $\mathbf{v}^{\mathsf{T}}\mathbf{x} = \mathbf{v}^{\mathsf{T}}\boldsymbol{\xi}$
- these are the points which have a fixed inner product (say $c = \mathbf{v}^{\mathsf{T}}\boldsymbol{\xi}$) with the vector $\mathbf{v}$
- let $\mathbf{v}$ be a unit vector in $\mathbb{R}^3$
  - then $\mathbf{v}^{\mathsf{T}}\mathbf{x} = ||\mathbf{x}||_2 \cos\ \theta = c, \quad \theta$ is the angle between $\mathbf{v}$ and $\mathbf{x}$
  - interpret geometrically those $\mathbf{x}$ such that $||\mathbf{x}||_2 \cos\ \theta = c$
  - determine the distance from the plane to the origin

Exercise

- if $\mathbf{v} = (1, 2, 3)$ and $c = 3$, determine the distance from the plane to the origin

Cartesian Equation of a Plane in $\mathbb{R}^3$

- recall: $\mathbf{v} \in \mathbb{R}^3$ (nonzero), $\mathbf{v}^\mathsf{T}\mathbf{x} = c$ characterizes a plane in $\mathbb{R}^3$
- $\mathbf{v}^\mathsf{T}\mathbf{x} = c$ yields

$$v_1x_1 + v_2x_2 + v_3x_3 = c \tag{3}$$

- (3) is called the Cartesian equation of the plane

Exercise

- a plane passes through the points $(1, 2, 3)$, $(3, 1, 2)$, and $(2, 3, 1)$
  - find the Cartesian equation of the plane
  - what is the distance from the plane to the origin ?
- what is the angle between the two planes characterized by $2x_1 - x_2 + x_3 = 6$ and $x_1 + 5x_2 + 3x_3 = 2$

Parametric Equation of a Plane in $\mathbb{R}^3$

- let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$
- suppose $\mathbf{u}$ and $\mathbf{v}$ are not parallel

Parametric Equation of a Plane in $\mathbb{R}^3$

- let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ be nonzero vector in $\mathbb{R}^3$
- identify $s\mathbf{u} + t\mathbf{v}$ if s and t are scalars, i.e., $s, t \in \mathbb{R}$
- identify the set $\{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$

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  - all points on the plane containing $\mathbf{u}$ and $\mathbf{v}$ (sketch)

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- identify $s\mathbf{u} + t\mathbf{v}$ if s and t are scalars, i.e., $s, t \in \mathbb{R}$
- identify the set $\{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$
  - all points on the plane containing $\mathbf{u}$ and $\mathbf{v}$ (sketch)
- let $\boldsymbol{\xi}$ be a vector in $\mathbb{R}^3$
- identify the set $\{\boldsymbol{\xi} + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$
  - all points on the plane which passes through $\boldsymbol{\xi}$ containing both vectors $\mathbf{u}$ and $\mathbf{v}$ (sketch)

Parametric Equation of a Plane in $\mathbb{R}^3$

- the parametric equation of the plane through $\boldsymbol{\xi}$ containing both vectors $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{x} = \boldsymbol{\xi} + s\mathbf{u} + t\mathbf{v}, \quad (s, t \in \mathbb{R}) \tag{4}$$

- (4) is also known as the vector equation of the plane
- each pair of the parameters (s, t) yields a unique point on the plane and vice versa

The Vector Product in $\mathbb{R}^3$

- recall the parametric equation of the plane aforementioned

$$\mathbf{x} = \boldsymbol{\xi} + s\mathbf{u} + t\mathbf{v}$$

- derive the corresponding Cartesian equation of the plane:

The Vector Product in $\mathbb{R}^3$

- recall the parametric equation of the plane aforementioned

$$\mathbf{x} = \boldsymbol{\xi} + s\mathbf{u} + t\mathbf{v}$$

- derive the corresponding Cartesian equation of the plane:

$$(u_2v_3 - u_3v_2)x_1 + (u_3v_1 - u_1v_3)x_2 + (u_1v_2 - u_2v_1)x_3 = c,$$

where
$c = (u_2v_3 - u_3v_2)\xi_1 + (u_3v_1 - u_1v_3)\xi_2 + (u_1v_2 - u_2v_1)\xi_3$

The Vector Product in $\mathbb{R}^3$

- from the Cartesian equation of the plane we find a normal vector $\mathbf{w}$ to the plane:

The Vector Product in $\mathbb{R}^3$

- from the Cartesian equation of the plane we find a normal vector $\mathbf{w}$ to the plane:

$$\mathbf{w} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \tag{5}$$

- $\mathbf{w}$ is called the vector product of vectors $\mathbf{u}$,$\mathbf{v} \in \mathbb{R}^3$
- the vector product of $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \times \mathbf{v}$, i.e., $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

The Vector Product to Characterize a Plane

- consider the plane which passes through $\boldsymbol{\xi}$ containing both vectors $\mathbf{u}$ and $\mathbf{v}$
- the plane is compactly characterized by

$$(\mathbf{u} \times \mathbf{v})^{\mathsf{T}}(\mathbf{x} - \boldsymbol{\xi}) = 0$$

- verify that $||\mathbf{u} \times \mathbf{v}||_2 = ||\mathbf{u}||_2||\mathbf{v}||_\mathbf{2} \sin\ \theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

Appendix

Mathematical Statement

- also called proposition [3]
- it is a sentence that is either true or false
  - 5 is odd is a math. statement which is true
  - 5 is even is a math. statement which is false
  - 5 is cool is not a math. statement

[3] For more details, see § 2.1, Hammack-2009

Implications

- many math. statements are written in the form [4]

$$\text{If } A \text{ Then } B, \tag{6}$$

  where A, B both are again math. statements

- the staement in (6) is equivalent to
  - If A is true Then B is true
  - $A \implies B$
  - A is sufficient for B
  - B is necessary for A
  - A only if B
  - not $B \implies$ not A [contrapositive of (6)]

[4] For more details, see § 2.3, Hammack-2009

Converse

- converse of $A \implies B$ is

$$B \implies A$$

- suppose we have both $A \implies B$ and $B \implies A$, this means..
  - $A \iff B$
  - A if and only if B
  - A is necessary and sufficient for B
  - B is necessary and sufficient for A

Proof Methods

- let us list 3 most popular ones, in particular for showing $A \implies B$
  - direct method
  - proof by contraposition
  - proof by contradiction
- see § 1.1 of Chong-2013
- for more details, see § 4,5,6 of Hammack-2009

Proof Methods

- consider the math. statement

$$N \in \mathbb{R} \text{ is divisible by } 4 \implies N \in \mathbb{R} \text{ is divisible by } 2 \quad (7)$$

- such implications are also called conditional statements
- exercise: prove (7) by [5]
  - direct method
  - proof by contraposition
  - proof by contradiction

[5] More easy to understand explanations are available in § 4,5,6 of Ref-4.

Set Definitions

- a set is usually defined using notation as follows: [6]

$$\mathcal{S} = \{x^2 \mid x = 2k, k \in \mathbb{N}\}$$

- $\mathcal{S}$ is the set of all things of the form x^2 such that $x = 2k$ and k is a natural number

- the brace means 'the set of all things of the form'

- vertical line '$|$' means 'such that'; the comma means 'and'

- of course, many more on x can be imposed by adding them with commas, e.g.,

$$\mathcal{S} = \{x \mid x \in \mathbb{R},\ x \geq 0,\ x \leq 5\}$$

[6] For more details, see § 1 of Hammack-2009

Set Definitions

- consider the following sets in $\mathbb{R}^2$

$$\mathcal{S}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$
$$\mathcal{S}_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$$
$$\mathcal{S}_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 2)^2 = 1\}$$
$$\mathcal{S}_5 = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 2)^2 = 1, 2x_1 + x_2 = 1\}$$
$$\mathcal{S}_6 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 2)^2 = 4, x_1 \geq 1\}$$
$$\mathcal{S}_7 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 2)^2 = 4, x_1 \geq 10\}$$

- exercise: mark those sets in $\mathbb{R}^2$

Quantifiers

- two heavily used symbols in mathematical statements
  - universal quantifier: $\forall$
  - existential quantifier: $\exists$
- '$\forall$' stands for the phrase "for all" or "for every"
- '$\exists$' stands for the phrase "there exists a" or "there is a"
- for more details, see § 2.7 of of Hammack-2009