Optimization of Communication Systems
EE 669

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Lecture 2: Vector Algebra (Part I)

Vectors

Vectors

- a vector is an ordered finite list of numbers
- vectors are usually written as vertical arrays, surrounded by square or curved brackets, e.g.,

$$\mathbf{a} = \begin{bmatrix} -1.5 \\ 1.0 \\ 3.2 \\ 0.0 \end{bmatrix} \quad \text{or} \quad \mathbf{a} = \begin{pmatrix} -1.5 \\ 1.0 \\ 3.2 \\ 0.0 \end{pmatrix}$$

- we may also write the vectors as numbers separated by commas and surrounded by parentheses, i.e., $\mathbf{a} = (-1.5, 1.0, 3.2, 0.0)$
- numbers in a vector are called entries or elements

n-Vectors

- a vector of size (length) n is called an n-vector (or n-tuple)
- 1-vector is considered to be the same as a number, e.g., $[1.5] = 1.5$
- some special vectors
  - zero vector, $\mathbf{0}$: is a vector with all elements equal to zero
  - ones vector, $\mathbf{1}$: is a vector with all its elements equal to one
  - standard unit vector, $\mathbf{e}_i$: is a vector whose all elements equal to zero, except the ith element which is one

Symbols to Denote Vectors

- typically used conventions: $\mathbf{a}$, $\vec{a}$
- ith element of n-vector $\mathbf{a}$ is denoted a_i
- $\mathbf{a}$ and $\mathbf{b}$ of the same size are equal (i.e., $\mathbf{a} = \mathbf{b}$), if $a_i = b_i$ for all i
- some other useful notation:
  - the set of all real numbers is written as $\mathbb{R}$
  - the set of all real n-vectors is denoted $\mathbb{R}^n$
  - $\mathbf{a} \in \mathbb{R}^n$ means that $\mathbf{a}$ is an element of the set $\mathbb{R}^n$

Examples of Vectors in Engineering

- see [pp. 6-11, Boyd-2018]

Position and Displacement

- a 2-vector can be used to represent either the position of a point or a displacement in a 2-dimensional (2-D) space, i.e., a plane
- a 3-vector can be used to represent either the position of a point or a displacement in a 3-D space, i.e., a plane
- a vector can also be used to represent the velocity or acceleration, at a given time, of a point that moves in a plane or 2-D or 3-D space
- illustrate the idea in a Cartesian plane, *cf.* [Fig. 1.1, Boyd-2018]

Displacement Vectors

- thinking of a vector as a displacement, the arrow we use to represent the vector need not have its initial point at the origin
- all arrows with the same length and direction, wherever they are located, represent the same displacement vector
- illustrate the idea in a Cartesian plane

Vector Addition

Vector Addition

- n-vectors $\mathbf{a}$ and $\mathbf{b}$ and can be added, the sum is denoted by $\mathbf{a} + \mathbf{b}$
- vector sum is by adding corresponding entries

$$\begin{bmatrix} -1.5 \\ 1.0 \\ 3.2 \\ 0.0 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1.0 \\ 0.8 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 2.0 \\ 4.0 \\ 0.5 \end{bmatrix}$$

- subtraction is similarly defined

Properties of Vector Addition

- $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ of the same size
- commutative: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- associative: $\mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ [1]
- $\mathbf{a} - \mathbf{a} = \mathbf{0}$, the size of $\mathbf{0}$ is the size of $\mathbf{a}$

[1] The size of the $\mathbf{0}$ (zero vector) follows from the context.

Addition Displacements

- if 2-vectors $\mathbf{a}$ and $\mathbf{b}$ are displacements, $\mathbf{a} + \mathbf{b}$ is the sum displacement
- same applies for 3-vectors $\mathbf{a}$ and $\mathbf{b}$ if they represent displacements
- illustrate the idea in a Cartesian plane *cf.* [Fig. 1.6, Boyd-2018]

Addition a Position to a Displacement

- what if a 2-vector $\mathbf{p}$ represents a position and a 2-vector $\mathbf{b}$ represents a displacement ?
- then $\mathbf{p} + \mathbf{a}$ is the position of the point $\mathbf{p}$, displaced by $\mathbf{a}$
- same applies for 3-vectors $\mathbf{p}$ and $\mathbf{a}$ of the same form
- illustrate the idea in a Cartesian plane *cf.* [Fig. 1.7, Boyd-2018]

Displacement between two Positions

- suppose 2-vectors $\mathbf{p}$ and $\mathbf{q}$ represent positions of two points
- then $\mathbf{p} - \mathbf{q}$ is the displacement vector from $\mathbf{q}$ to $\mathbf{p}$
- same applies for 3-vectors $\mathbf{p}$ and $\mathbf{q}$
- illustrate the idea in a Cartesian plane *cf.* [Fig. 1.8, Boyd-2018]

Scalar Vector Multiplication

Scalar Vector Multiplication

- scalar α and an n-vector $\mathbf{a}$ can be multiplied, i.e.,

$$\alpha\mathbf{a} = (\alpha a_1, \ldots, \alpha a_n)$$

- also denoted $\mathbf{a}\alpha$
- some properties
  - associative: $(\beta\alpha)\mathbf{a} = \beta(\alpha\mathbf{a})$
  - left distributive: $(\beta + \alpha)\mathbf{a} = \beta\mathbf{a} + \alpha\mathbf{a}$
  - right distributive: $\beta(\mathbf{a} + \mathbf{b}) = \beta\mathbf{a} + \beta\mathbf{b}$
- illustrate the idea in a Cartesian plane *cf.* [Fig. 1.10, Boyd-2018]

Linear Combinations

- for vectors $\mathbf{a}_1, \ldots, \mathbf{a}_1$ and scalars $\beta_1, \ldots, \beta_m$,

$$\beta_1 \mathbf{a}_1 + \cdots + \beta_m \mathbf{a}_m$$

  is a linear combination of the vectors

- $\beta_1, \ldots, \beta_m$ are the coefficients

- note that for any n-vector $\mathbf{b}$, we have

$$\mathbf{b} = b_1 \mathbf{e}_1 + \cdots + b_n \mathbf{e}_n$$

- illustrate the idea in a Cartesian plane

Inner Product

Inner Product

- product of two vectors of same size make no sense
- however a scalar quantity called inner product is defined
- the inner product (or dot product) of n-vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^{\mathsf{T}}\mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- properties:
  - Commutativity: $\mathbf{a}^{\mathsf{T}}\mathbf{b} = \mathbf{b}^{\mathsf{T}}\mathbf{a}$
  - Associativity with scalar multiplication:

$$(\gamma\mathbf{a})^{\mathsf{T}}\mathbf{b} = \gamma(\mathbf{a}^{\mathsf{T}}\mathbf{b})$$

  - Distributivity with vector addition: $(\mathbf{a} + \mathbf{b})^{\mathsf{T}}\mathbf{c} = \mathbf{a}^{\mathsf{T}}\mathbf{c} + \mathbf{b}^{\mathsf{T}}\mathbf{c}$

Inner Product: Examples

- picking ith entity of an n-vector $\mathbf{a}$: $a_i = \mathbf{e}_i^\mathsf{T}\mathbf{a}$
- sum of entries of an n-vector $\mathbf{a}$: $\sum_{i=1}^n a_i = \mathbf{1}^\mathsf{T}\mathbf{a}$
  - here the size of ones vector is to be determined from the context.
- sum of squares of entries of an n-vector $\mathbf{a}$: $\sum_{i=1}^n a_i^2 = \mathbf{a}^\mathsf{T}\mathbf{a}$

Norm

Euclidean Norm of a Vector

- the Euclidean norm (or ℓ_2 norm) of an n-vector $\mathbf{x}$ is

$$||\mathbf{x}||_2 = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- used to measure the size of a vector
- reduces to absolute value for $n = 1$
- illustrate the idea in a Cartesian plane

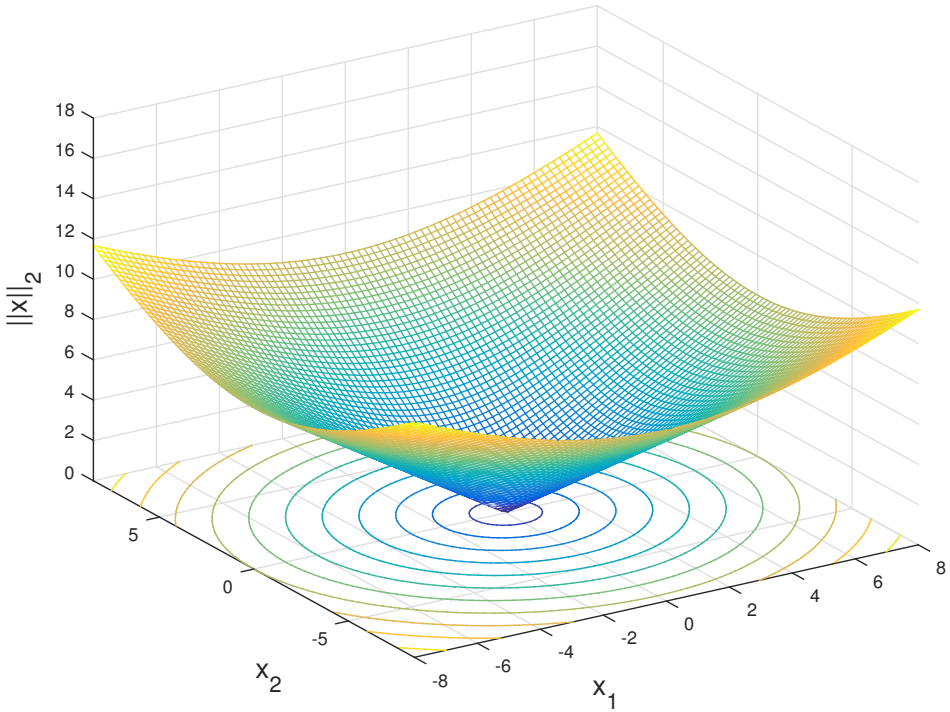
Euclidean Norm to Define Unit Vectors

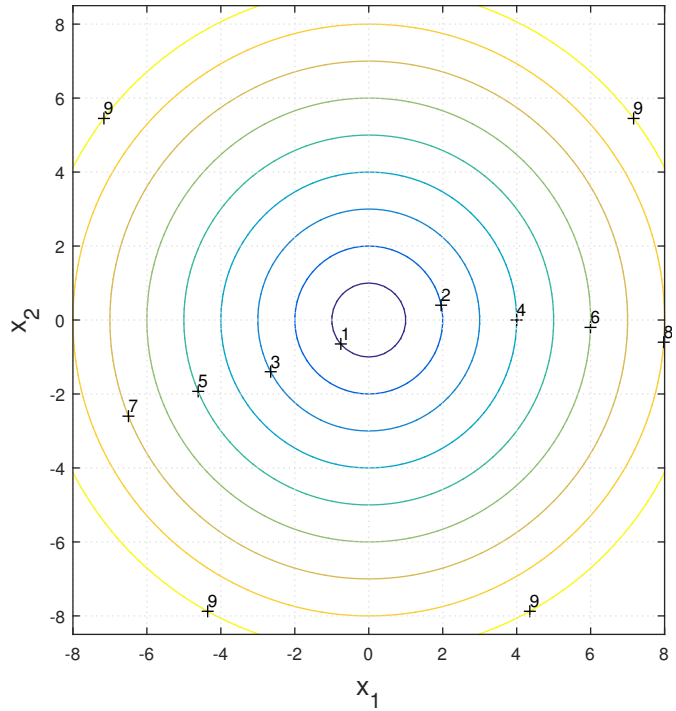
- vectors $\mathbf{u}$ with $||\mathbf{u}||_2 = 1$ are called unit vectors
- given an vector $\mathbf{x} \neq \mathbf{0}$, the unit vector along $\mathbf{x}$ is given by

$$\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||_2}$$

- illustrate the idea in a Cartesian plane

Graph of ℓ_2 Norm





- graph of $||\mathbf{x}||_2$ and contours of $||\mathbf{x}||_2$

Properties of Euclidean Norm

- for any n-vectors $\mathbf{x}$ and $\mathbf{y}$, and any scalar β
  - Nonnegativity, i.e.,
$$||\mathbf{x}||_2 \geq 0$$
  - Definiteness, i.e.,
$$||\mathbf{x}||_2 = 0 \implies \mathbf{x} = \mathbf{0}$$
  - homogeneity, i.e.,
$$||\beta\mathbf{x}||_2 = |\beta| \ ||\mathbf{x}||_2$$
  - triangle inequality , i.e.,
$$||\mathbf{x} + \mathbf{y}||_2 \leq ||\mathbf{x}||_2 + ||\mathbf{y}||_2$$

Other Norms

- ℓ_1 norm:

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i|$$

- ℓ_∞ norm:

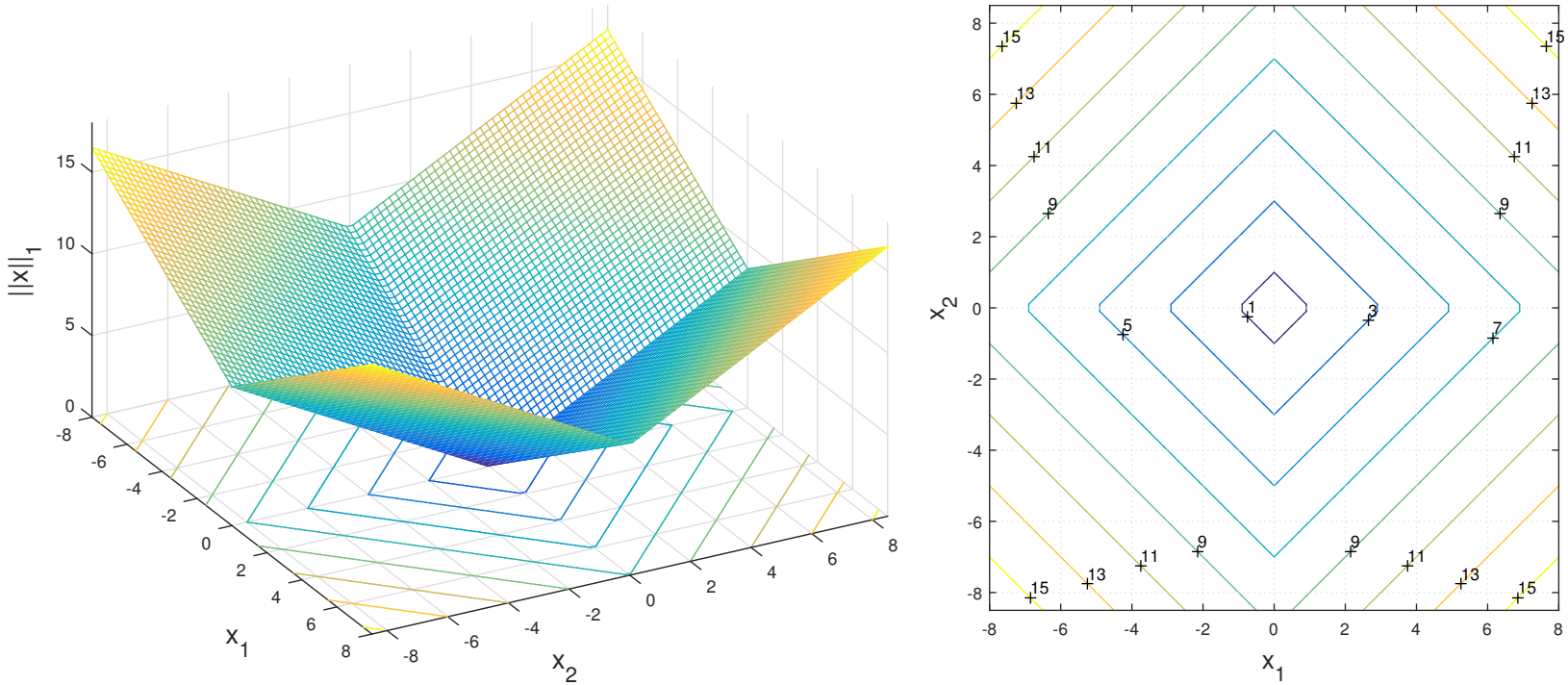
$$||\mathbf{x}||_\infty = \max_{i \in \{1,\ldots,n\}} |x_i| = \max_i |x_i|$$

- ℓ_p norm:

$$||\mathbf{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad p \geq 1$$

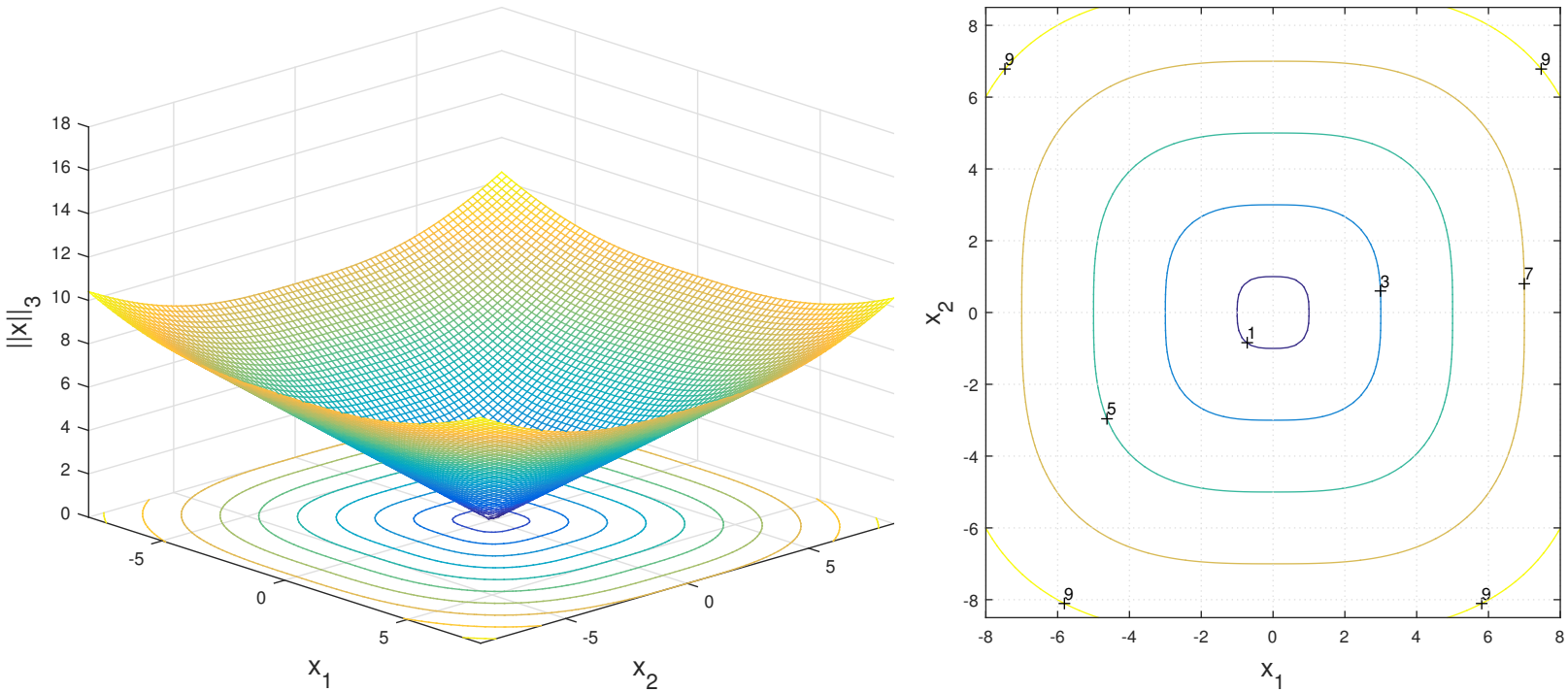
- note: all properties of ℓ_2 norm also hold for ℓ_1, ℓ_∞, and ℓ_p, *cf.* p. 20

Graph of ℓ_1 Norm



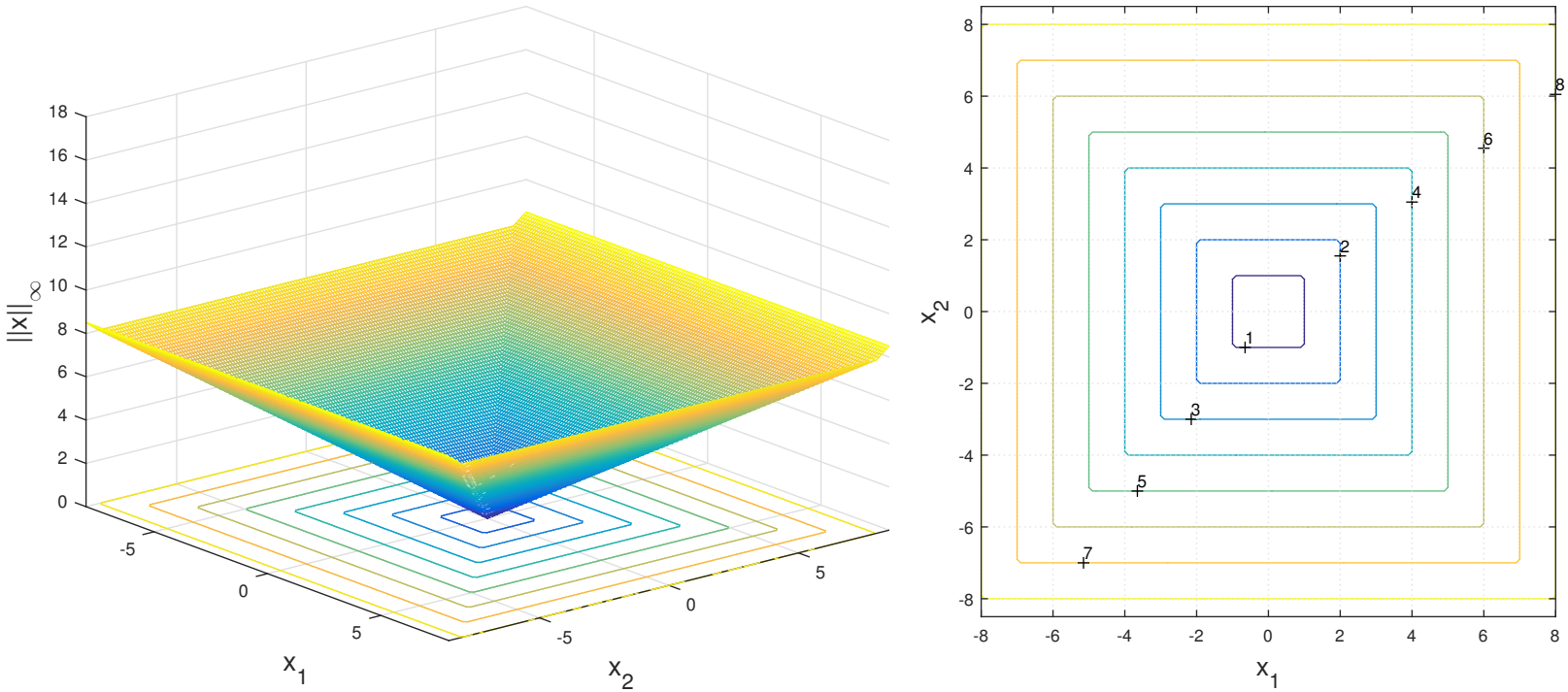
- graph of $||\mathbf{x}||_1$ and contours of $||\mathbf{x}||_1$

Graph of ℓ_p Norm



- graph of $||\mathbf{x}||_p$ and contours of $||x||_p$ with $p = 3$

Graph of ℓ_∞ Norm



- graph of $||\mathbf{x}||_\infty$ and contours of $||\mathbf{x}||_\infty$

Distance

Distance Between Position Vectors

- the Euclidean distance between two points represented by n-vectors $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$||\mathbf{x}-\mathbf{y}||_2 = \sqrt{(\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y})} = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

- agrees with ordinary distance for $n = 1, 2, 3$
- illustrate the idea in a Cartesian plane

Triangle Inequality

- triangle with vertices at positions $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$
- the lengths of edges are $||\mathbf{a}-\mathbf{b}||_2$,$||\mathbf{b}-\mathbf{c}||_2$, and $||\mathbf{a}-\mathbf{c}||_2$
- triangle inequality: the third edge length is no longer than sum of other two
- i.e.,

$$||\mathbf{a}-\mathbf{c}||_2 = ||(\mathbf{a}-\mathbf{b})+(\mathbf{b}-\mathbf{c})||_2 \leq ||\mathbf{a}-\mathbf{b}||_2 + ||\mathbf{b}-\mathbf{c}||_2$$

- illustrate the idea in a Cartesian plane

Angle

Cauchy-Schwarz Inequality

- for two n-vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a}^\mathsf{T}\mathbf{b}| \leq ||\mathbf{a}||_2||\mathbf{b}||_2$$

- i.e.,

$$|a_1b_1 + \cdots +_n b_n| \leq \sqrt{a_1^2 + \cdots + a_n^2}\sqrt{b_1^2 + \cdots + b_n^2}$$

- Cauchy-Schwarz can be used to show the triangle inequality
  - start with $||\mathbf{x} + \mathbf{y}||_2^2$

Cauchy-Schwarz Inequality

- interpret Cauchy-Schwarz inequality in a Cartesian plane
- Derive the Cauchy-Schwarz inequality
  - true if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$
  - when both $\mathbf{a}$ and $\mathbf{b}$ are nonzero, we can start by
    - let $\alpha = ||\mathbf{a}||_2$ and $\beta = ||\mathbf{b}||_2$
    - we have $0 \leq ||\beta\mathbf{a} - \alpha\mathbf{b}||_2$, expand the RHS
    - divide by $2\alpha\beta$ to get $\mathbf{a}^\mathsf{T}\mathbf{b} \leq ||\mathbf{a}||_2||\mathbf{b}||_2$
    - apply $-\mathbf{a}$ to get the other half of the inequality

Angle Between Vectors in 2-D and 3-D

- cosine rule states (draw the corresponding triangle)

$$c^2 = a^2 + b^2 - 2ab\cos\ \theta$$

- rewrite the cosine rule in terms of vectors to yield

$$||\mathbf{v} - \mathbf{w}||_2^2 = ||\mathbf{v}||_2^2 + ||\mathbf{w}||_2^2 - 2||\mathbf{v}||_2||\mathbf{w}||_2\cos\ \theta \qquad (1)$$

- expand $||\mathbf{v} - \mathbf{w}||_2^2$ to yield

$$||\mathbf{v} - \mathbf{w}||_2^2 = ||\mathbf{v}||_2^2 + ||\mathbf{w}||_2^2 - 2\mathbf{v}^\mathsf{T}\mathbf{w} \qquad (2)$$

Angle Between Vectors in 2-D and 3-D

- thus from (1) and (2), we have $\mathbf{v}^\mathsf{T}\mathbf{w} = ||\mathbf{v}||_2||\mathbf{w}||_2 \cos\ \theta$
- so θ is given by

$$\cos\theta = \frac{\mathbf{v}^\mathsf{T}\mathbf{w}}{||\mathbf{v}||_2||\mathbf{w}||_2}, \quad 0 \leq \theta \leq \pi$$

- find the angle between vectors $(2, 0)$ and $(-1, 1)$

Angle Between Vectors in 2-D and 3-D

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- so θ is given by

$$\cos\theta = \frac{\mathbf{v}^{\mathsf{T}}\mathbf{w}}{||\mathbf{v}||_2||\mathbf{w}||_2}, \quad 0 \leq \theta \leq \pi$$

- find the angle between vectors $(2, 0)$ and $(-1, 1)$, $\theta = -3\pi/4$

Classification of Angles

- let θ be the angle between 2- or 3-vectors $\mathbf{v}$ and $\mathbf{w}$
  - $\theta = \pi/2$: $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, $(\mathbf{v}^\mathsf{T}\mathbf{w} = 0)$
  - $\theta = 0$: $\mathbf{v}$ and $\mathbf{w}$ are aligned, $(\mathbf{v}^\mathsf{T}\mathbf{w} = ||\mathbf{v}||_2||\mathbf{w}||_2)$
  - $\theta = \pi$: $\mathbf{v}$ and $\mathbf{w}$ are anti-aligned, $(\mathbf{v}^\mathsf{T}\mathbf{w} = -||\mathbf{v}||_2||\mathbf{w}||_2)$
  - $\theta \leq \pi/2$: $\mathbf{v}$ and $\mathbf{w}$ make an acute angle, $(\mathbf{v}^\mathsf{T}\mathbf{w} \geq 0)$
  - $\theta \geq \pi/2$: $\mathbf{v}$ and $\mathbf{w}$ make an obtuse angle, $(\mathbf{v}^\mathsf{T}\mathbf{w} \leq 0)$
- illustrate the idea in a Cartesian plane

Angle Between n-Vectors

- angle $\theta_{\mathbf{vw}}$ between two nonzero n-vectors $\mathbf{v}$, $\mathbf{w}$ is defined as

$$\theta_{\mathbf{vw}} = \arccos\left(\frac{\mathbf{v}^{\mathsf{T}}\mathbf{w}}{||\mathbf{v}||_2||\mathbf{w}||_2}\right)$$

- $\theta_{\mathbf{vw}}$ is the number in $[0, \pi]$ that satisfies

$$\mathbf{v}^{\mathsf{T}}\mathbf{w} = ||\mathbf{v}||_2||\mathbf{w}||_2 \cos\theta_{\mathbf{vw}}$$

- the above definition coincides with the ordinary angle between vectors in 2-D and 3-D, *cf.* pp. 29-31

Orthogonal n-Vectors

- similar to the case of 2- or 3-vectors, n-vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be orthogonal to each other whenever

$$\mathbf{v}^{\mathsf{T}}\mathbf{w} = 0$$

- of course, visual concept of right angle is not at our disposal in higher Euclidean space