Optimization of Communication Systems
EE 669

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Lecture 12: Applications II [1]

[1] The content in this lecture is based on § 13,§ 15, § 16 of [Calafiore-2014].

Filter Designs

Amplitude Response of Low-Pass Filters

- e.g., linear-phase filters
- the amplitude response: type-I linear-phase filter

$$A(\theta) = h_N + \sum_{k=0}^{N-1} h_k \cos\left((N-k)\theta\right) \triangleq a^{\mathsf{T}}(\theta) h$$

where h_k, $k = 0, 1, \ldots, 2N$ are the filter coefficients with midpoint symmetry

- task: determine $h = (h_0, h_1, \cdots, h_N) \in \mathbb{R}^{N+1}$ to yield an specified shape of the amplitude response

Associated Constraints

- stop-band specs:

$$-\delta_s \leq A(\theta) \leq \delta_s, \quad \theta \in [\theta_s, \pi]$$

- pass-band specs:

$$1 - \delta_p \leq A(\theta) \leq 1 + \delta_p, \quad \theta \in [0, \theta_p]$$

Frequency Discretization

- stop-band specs:

$$-\delta_s \leq A(\theta_i) = a^{\mathsf{T}}(\theta_i)h \leq \delta_s, \quad i = 1, 2, \ldots, N_s$$

- pass-band specs:

$$1 - \delta_p \leq A(\theta_i) = a^{\mathsf{T}}(\theta_i)h \leq 1 + \delta_p, \quad i = N_s + 1, \ldots, N_s + N_p$$

Compute h via LP

- compute a feasible h

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & -\delta_s \leq a^{\mathsf{T}}(\theta_i)h \leq \delta_s,\ i = 1, 2, \ldots, N_s \\ & 1 - \delta_p \leq a^{\mathsf{T}}(\theta_i)h \leq 1 + \delta_p,\ i = N_s + 1, \ldots, N_s + N_p \end{array}$$

Compute h via LP

- minimize stop-band attenuation

$$
\begin{array}{ll}
\text{minimize} & \delta_s \\
\text{subject to} & -\delta_s \leq a^{\mathsf{T}}(\theta_i)h \leq \delta_s,\ i = 1, 2, \ldots, N_s \\
& 1 - \delta_p \leq a^{\mathsf{T}}(\theta_i)h \leq 1 + \delta_p,\ i = N_s + 1, \ldots, N_s + N_p
\end{array}
$$

Compute h via LP

- minimize pass-band attenuation

$$
\begin{array}{ll}
\text{minimize} & \delta_p \\
\text{subject to} & -\delta_s \leq a^{\mathsf{T}}(\theta_i)h \leq \delta_s,\ i = 1, 2, \ldots, N_s \\
& 1 - \delta_p \leq a^{\mathsf{T}}(\theta_i)h \leq 1 + \delta_p,\ i = N_s + 1, \ldots, N_s + N_p
\end{array}
$$

Compute h via Reference Matching

- a reference design $\tilde{A}_{\mathrm{ref}}(\theta)$ is given
- minimize the 'error' due to mismatch between $\tilde{A}_{\mathrm{ref}}(\theta)$ and $A(\theta)$
- depending on the way we compute the error (loss), the objectibe function varies, e.g.,
  - LS design
  - Chebyshev design

LS Design

- minimize the sum of error-squared due to mismatch between $\tilde{A}_{\text{ref}}(\theta)$ and $A(\theta)$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{M} \left(A(\theta_i) - \tilde{A}_{\text{ref}}(\theta_i) \right)^2 \\ \text{subject to} & h \in \mathbb{R}^{N+1} \end{array}$$

LS Design

- recall: $A(\theta_i) = a^{\mathsf{T}}(\theta_i)h$
- thus, the above problem is a LS problem:

$$\begin{array}{ll} \text{minimize} & ||Bh - b||^2 \\ \text{subject to} & h \in \mathbb{R}^{N+1} \end{array}$$

where $B \in \mathbb{R}^{M \times N+1}$ and $b \in \mathbb{R}^M$ are known matrices

Chebyshev Design

- minimize the max of absolute error due to mismatch between $\tilde{A}_{\mathrm{ref}}(\theta)$ and $A(\theta)$

$$\begin{array}{ll} \text{minimize} & \max\limits_{i\in\{1,2,\ldots,M\}} |A(\theta_i) - \tilde{A}_{\mathrm{ref}}(\theta_i)| \\ \text{subject to} & h \in \mathbb{R}^{N+1} \end{array}$$

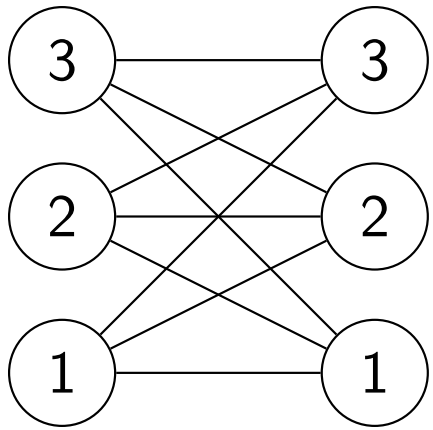
Chebyshev Design

- recall: $A(\theta_i) = a^{\mathsf{T}}(\theta_i)h$
- thus, the above problem in its epigraph form is:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a^{\mathsf{T}}(\theta_i)h - \tilde{A}_{\text{ref}}(\theta_i) \leq t, \;\; i = 1, 2, \ldots, M \\ & -a^{\mathsf{T}}(\theta_i)h + \tilde{A}_{\text{ref}}(\theta_i) \leq t, \;\; i = 1, 2, \ldots, M \end{array}$$

Matching

Weighted Bipartite Matching



- n agents are to be assigned to n tasks, in a one-to-one fashion
- cost of matching agent i to task j is w_{jj}
- define the variables:

$$x_{ij} = \begin{cases} 1 & \text{agent } i \text{ is assigned to } j \\ 0 & \text{otherwise} \end{cases}$$

Weighted Bipartite Matching

- the assignment which yields the minimum cost:

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_{ij} \\
\text{subject to} & x_{ij} \in \{0,1\} \qquad i,j = 1,\ldots,n \\
& \sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,\ldots,n \\
& \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1,\ldots,n
\end{array}
$$

Shortest Path Problem

Shortest Path Problem

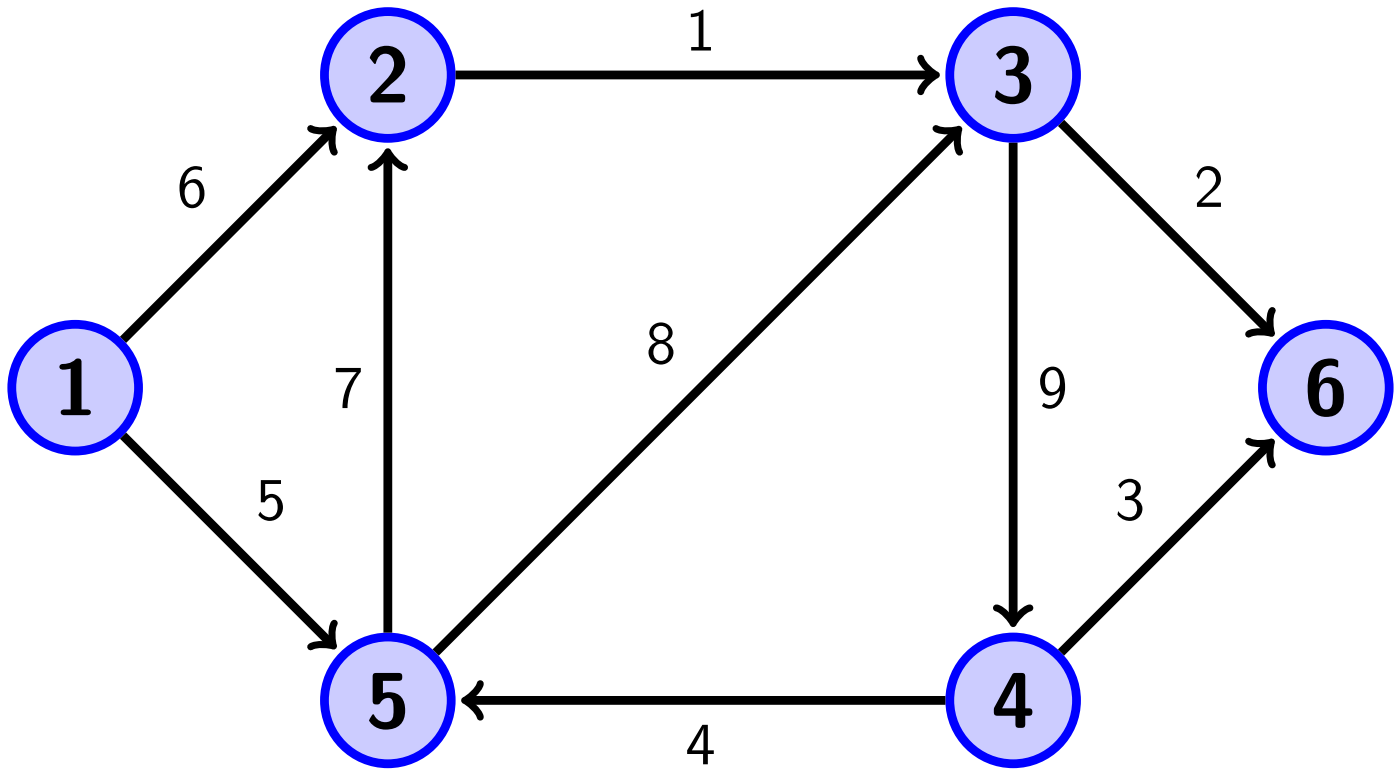
- a path between two vertices (or nodes) in a directed graph is to be found
- the sum of the weights along the edges in the path is should be minimum
- consider a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$
- the set of vertices $\mathcal{V} = \{1, \ldots, n\}$
- the set of edges is $\mathcal{E}$

Directed Incidence Matrix

- $A \in \mathbb{R}^{n \times |\mathcal{E}|}$: (directed) incidence matrix of the graph
- in particular, elements of A is given by

$$A_{ij} = \begin{cases} 1 & \text{edge } j \in \mathcal{E} \text{ starts at node } i \\ -1 & \text{edge } j \in \mathcal{E} \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

Directed Incidence Matrix



Directed Incidence Matrix

- A for the graph above is a 6×9 matrix
- in particular, we have

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= [a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \ a_8 \ a_9]$$

Paths

- paths from a source node to a destination node (without loops) can be characterized by using columns of A
- e.g., suppose source node is 1 and destination node is 6
- this configuration is uniquely characterized by vector $b = (1, 0, 0, 0, 0, -1)$

Paths

- consider a linear combination of the columns of A to create above b
- the coefficients of columns of A constitute a path from node 1 to 6, e.g.
  - one can go from node $1 \rightarrow 2 \rightarrow 3 \rightarrow 6$
  - i.e., edges $6 \rightarrow 1 \rightarrow 2$
  - create a 9-vector by indicating the above edges: $x = (1, 1, 0, 0, 0, 1, 0, 0, 0)$
  - note that $Ax = b$, i.e., a blend of columns yields the characterization of source and destination nodes given by b

Paths

- in other words,

$\exists$ a path from a source to a destination $\iff$

$\exists\ x \in \mathbb{R}^{|\mathcal{E}|}$ such that $Ax = b$

Shortest Path Problem

- thus the shortest path problem from source to destination characterized by $b \in \mathbb{R}^n$ is:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{|\mathcal{E}|} w_i x_i \\ \text{subject to} & Ax = b \\ & x_i \in \{0,1\}, \qquad i = 1, \ldots, |\mathcal{E}| \end{array}$$

where w_i denote the cost of travelling along edge i

SINR Maximization in Wireless Communication

SINR at Receivers

- n transmit/receive pairs
- power allocated at transmitter i to receiver i is p_i
- path gains from transmitter i to receiver j is g_{ji}
- SINR at receiver i is given by

$$\frac{g_{jj}p_j}{\sigma^2 + \sum_{i \neq j} g_{ji}p_i}$$

- the mimimum SINR threshold for uninterrupted communication is γ for all receivers

Optimize Sum Power

- task: compute the minimum sum power to yield the minimum SINR threshold at all the receivers

- the corresponding problem is:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} p_i \\ \text{subject to} & \dfrac{g_{jj}p_j}{\sigma^2 + \sum_{i\neq j} g_{ji}p_i} \geq \gamma, \qquad j = 1, \ldots, n \\ & p_i \geq 0, \qquad j = 1, \ldots, n \end{array}$$

- the above problem is a geometric program

Maximize the Minimum SINR

- task: maximize the minimum SINR if the power budget is P
- the corresponding problem is:

$$\begin{array}{ll} \text{maximize} & \min\limits_{i\in\{1,\ldots,n\}} \dfrac{g_{jj}p_j}{\sigma^2+\sum_{i\neq j} g_{ji}p_i} \\ \text{subject to} & \sum_{i=1}^n p_i \leq P \\ & p_i \geq 0, \qquad i=1,\ldots,n \end{array}$$

- the above problem is a geometric program! verify

Water-Filling in Parallel Gaussian Channel

Parallel Gaussian Channel

- n parallel Gaussian channels are given
- variance of the noise in ith channel is σ_i^2
- power allocation to ith channel is p_i
- total power budget is P
- what is the capacity of the parallel Gaussian channel?

Sum Capacity

- task: maximize the sum capacity given the power budget P
- the corresponding problem is:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} \log\left(1+\frac{p_i}{\sigma_i^2}\right) \\ \text{subject to} \quad & \sum_{i=1}^{n} p_i \leq P \\ & p_i \geq 0, \qquad i=1,\ldots,n \end{aligned}$$

- solution can be found n closed-form by writing KKT optimality conditions [2]
- the solution method is called the water-filling

[2] See [Boyd-2004], page 245.

Square-Root LASSO

Square-Root LASSO

- a little different compared to LASSO [3]
- the corresponding problem is:

$$\begin{array}{ll} \text{minimize} & ||\Phi w - b||_2 + \lambda ||w||_1 \\ \text{subject to} & w \in \mathbb{R}^2 \end{array}$$

- the above problem is a SOCP! verify

[3] See lecture slides set 11, page 8.

Facility Location Problem

Facility Location Problem

- task: locate a warehouse to serve a number of service locations
- service locations are given: $b_i \in \mathbb{R}^2$, $i = 1, \ldots, m$
- compute the location $x \in \mathbb{R}^2$ such that some cost criterion is minimized

Minimize the Maximum Distance

- one criterion is to minimize the maximum distance to service locations
- the corresponding problem is:

$$\begin{array}{ll} \text{minimize} & \max\limits_{i \in \{1,2,\ldots,m\}} ||x - b_i||_2 \\ \text{subject to} & x \in \mathbb{R}^2 \end{array}$$

- the above problem is a SOCP! verify

Minimize Average Transport Cost

- another criterion is to minimize the sum of the distances to service locations
- the corresponding problem is:

$$\begin{array}{ll} \text{minimize} & (1/m) \sum_{i=1}^{m} \; ||x - b_i||_2 \\ \text{subject to} & x \in \mathbb{R}^2 \end{array}$$

- the above problem is a SOCP! verify

Matrix Completion Problem

Matrix Completion

- task: recover a matrix X from incomplete information about its entries
- some a priori knowledge or assumptions are available, e.g.,
  - rank is small
  - NETFLIX problem
  - rows in X are users and columns are movies
  - entries represent (user,movie)-preferences

Matrix Completion

- the matrix X can be technically considered as low-rank [4]
- the corresponding problem is:

$$\begin{array}{ll} \text{minimize} & \operatorname{rank}(X) \\ \text{subject to} & X_{ij} = d_{ij} \end{array}$$

- the above problem is not convex

[4] See for details E. Candés, B. Recht, *Exact matrix completion via convex optimization*, Foundations of Computational Mathematics, 2009.

Matrix Completion

- a commonly used convex reformulation uses the nuclear norm instead of $\mathrm{rank}(\cdot)$ [5]
- the convex formulation is:

$$\begin{array}{ll} \text{minimize} & ||X||_* \\ \text{subject to} & X_{ij} = d_{ij} \qquad (i,j) \in \mathcal{J} \end{array}$$

- the above problem has an equivalent convex formulation

[5] See appendix of the lecture slides set 11.

Matrix Completion

- in fact, the following SDP is equivalent to the previous nuclear norm minimization
- the convex formulation is [6]:

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(Y) + \operatorname{tr}(Z) \\ \text{subject to} & \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \succeq 0 \\ & X_{ij} = d_{ij} \qquad (i,j) \in \mathcal{J} \end{array}$$

- where the variables are $X \in \mathbb{R}^{m \times n}$, $Y \in \mathbb{S}^n$, and $Z \in \mathbb{S}^n$

[6] This formulation is due to M. Fazel, *Matrix Rank Minimization with Applications*, Ph.D. thesis, Stanford University, 2002.