Traffic Engineering
EN3370

Chathuranga Weeraddana

December 2020

Lecture 7: Elementary Queuing Theory II

Places to Read

- see [Kleinrock, § 3]

Equilibrium Solution

Equilibrium Solution

- let us explore the limiting behaviour of these equations
- define [1]

$$p_k \triangleq \lim_{t \to \infty} P_k(t) \tag{1}$$

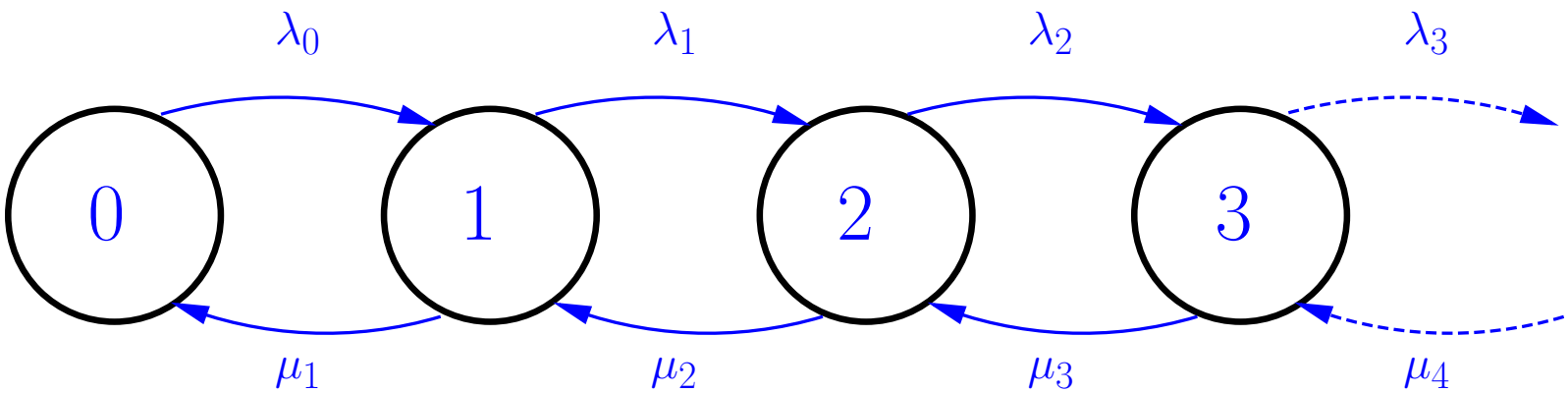
- p_k: the limiting probability that the system contains k members (or equivalently is in state k)
- let us assume the above limit exists and proceed [2]

[1] Here the notation is different to $\pi_k(t)$ and π_k used in lecture slides set 5, page 29, even though they carry the same meaning.

[2] See Appendix for necessary and sufficient conditions.

Birth-Death Queueing Systems in Equilibrium

Equilibrium Solution



Equilibrium Solution

- we can concisely present equations that define the equilibrium solution as

$$0 = -(\lambda_k + \mu_k)p_k + \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}, \ k \in \mathbb{Z} \qquad (2)$$

  where we adopt the convention that $\lambda_{-1} = \lambda_{-2} = \cdots = 0$, $\mu_0 = \mu_{-1} = \mu_{-2} = \cdots = 0$, and $p_{-1} = p_{-2} = \cdots = 0$

- in addition we have the conservation relation:

$$\sum_{k=0}^{\infty} p_k = 1 \qquad (3)$$

Computing the Solution

- recursion yields

$$p_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \, p_0 = p_0 \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}, \quad k = 0, 1, 2, \ldots \quad (4)$$

Computing the Solution

- you may validate the solution (4) by substituting it in (**??**)
- conservation relation (3) can be used to compute p_0, i.e.,

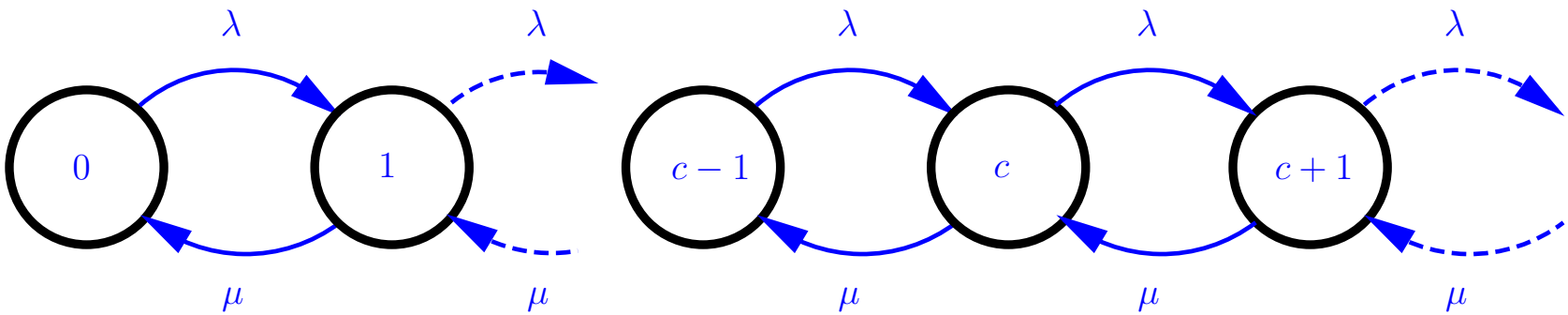
$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}} \tag{5}$$

- the equations (5) and (4) are used in the rest of this lecture

Elementary Queues

M/M/1, The Classical Queuing System

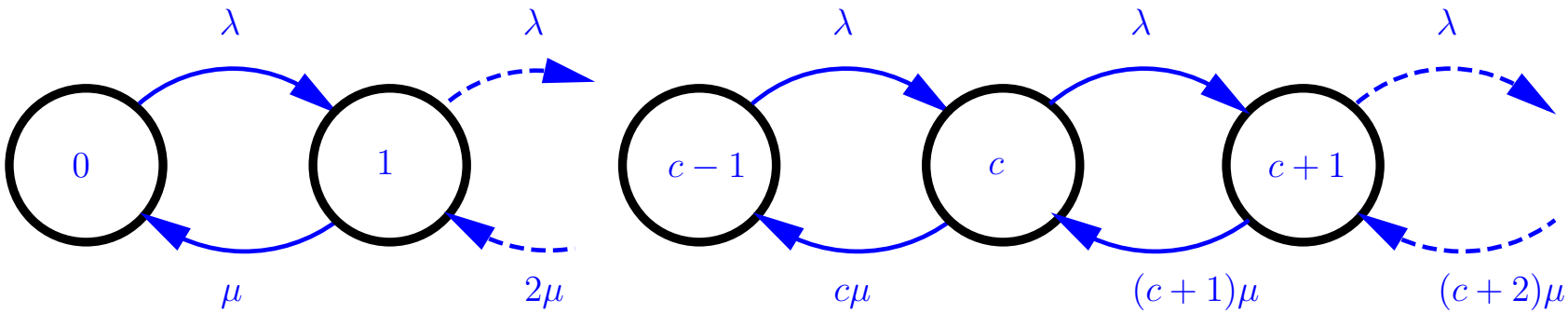
M/M/1



- we already discussed !

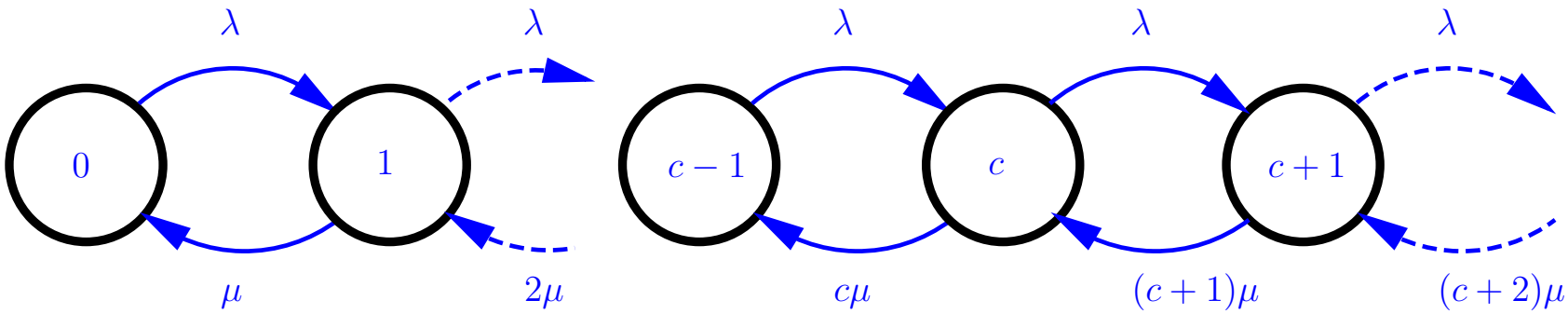
M/M/∞, Infinite Number od Servers (or Responsive Servers)

M/M/∞



- there are infinitely many servers
- another interpretation: the server is responsive
- unlimited waiting room

M/M/∞



- here we have

$$\lambda_k = \lambda \qquad k = 0, 1, \ldots$$
$$\mu_k = k\mu \qquad k = 1, 2, \ldots$$

Existence of Equilibrium Solution

- necessary and sufficient conditions for the existence of equilibrium probabilities $\{p_k\}$: [3]

$$\lambda < \mu$$

- let $\rho = \lambda/\mu$

[3] Recall the sums S_1 and S_2 in appendix of the previous lecture.

Equilibrium Solution

- from (4), we get

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu}$$
$$= p_0 \ \frac{1}{k!} \left(\frac{\lambda}{\mu}\right)^k , \qquad k \geq 0$$

- from (5), we get

$$p_0 = \frac{1}{\sum_{k=1}^{\infty} \ \frac{1}{k!} \left(\frac{\lambda}{\mu}\right)^k} = \exp\ (-\lambda/\mu)$$

Equilibrium Solution

- thus, we have

$$p_0 = e^{-\rho}$$

- moreover,

$$p_k = \frac{1}{k!}\rho^k e^{-\rho} \qquad k = 0, 1, 2, \ldots$$

Average of No. of Customers

- average number of customers $\bar{N}$:

$$\begin{aligned}\bar{N} &= \sum_{k=0}^{\infty} k p_k \\ &= \rho\end{aligned}$$

Variance of No. of Customers

- variance of the number of customers σ_N^2:

$$\begin{aligned}\sigma_N^2 &= \sum_{k=0}^{\infty}(k-\rho)^2 p_k \\ &= \rho\end{aligned}$$

Average Waiting Time

- let T denote the average time spent in the system
- λ: average customer arrival rate
- apply Little's law:

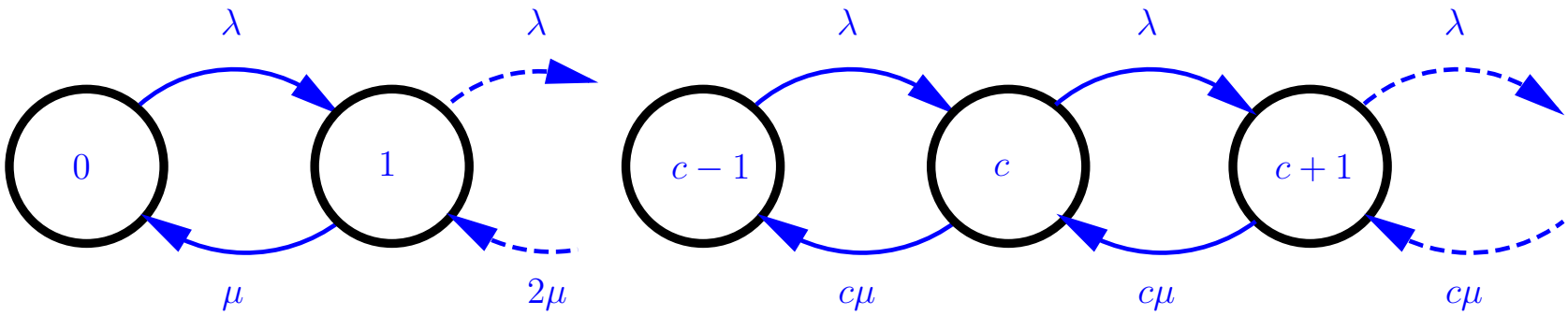
$$\bar{N} = \lambda \, T$$

- thus we have

$$T = \frac{\rho}{\lambda} = \frac{1}{\mu}$$

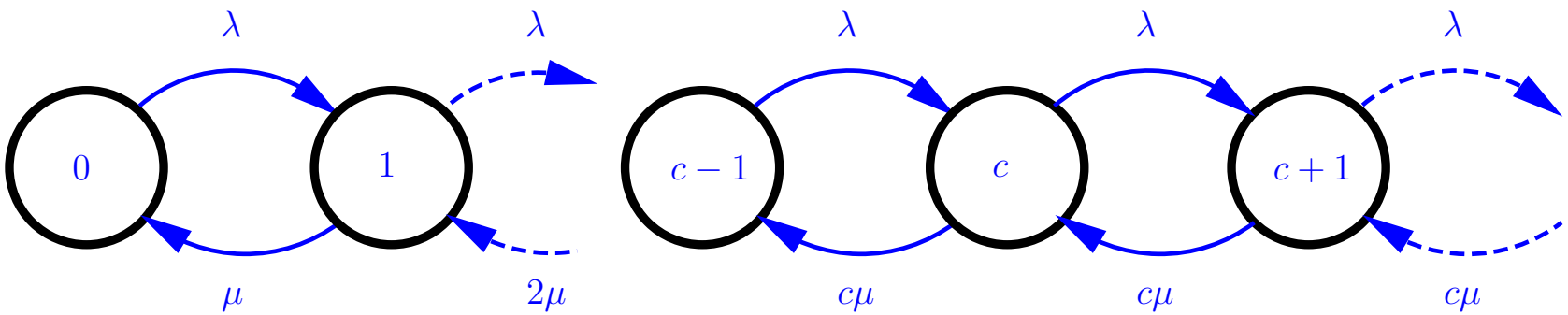
M/M/c, c Number of Servers

M/M/c



- there are c servers
- unlimited waiting room

M/M/c



- here we have

$$\lambda_k = \lambda \qquad k = 0, 1, \ldots$$

$$\mu_k = \begin{cases} k\mu & 0 \leq k \leq c \\ c\mu & k \geq c \end{cases}$$

Existence of Equilibrium Solution

- necessary and sufficient conditions for the existence of equilibrium probabilities $\{p_k\}$:

$$\lambda < c\mu$$

- let $\rho = \lambda/(c\mu)$

Equilibrium Solution

- from (4), for $k \leq c$ we get

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu}$$
$$= p_0 \frac{1}{k\,!} \left(\frac{\lambda}{\mu}\right)^k$$

- similarly, for $k > c$ we get

$$p_k = p_0 \prod_{i=0}^{c-1} \frac{\lambda}{(i+1)\mu} \prod_{j=c}^{k-1} \frac{\lambda}{c\mu}$$
$$= p_0 \frac{1}{c\,!\; c^{k-c}} \left(\frac{\lambda}{\mu}\right)^k$$

Equilibrium Solution

- further simplification yields

$$p_k = \begin{cases} p_0 \dfrac{(c\rho)^k}{k\,!} & k \leq c \\ \\ p_0 \dfrac{\rho^k m^m}{m\,!} & k \geq c \end{cases}$$

Equilibrium Solution

- from (5), we get

$$p_0 = \left[1 + \sum_{k=1}^{c-1} \frac{(c\rho)^k}{k\,!} + \sum_{k=c}^{\infty} \frac{(c\rho)^k}{c\,!}\ \frac{1}{c^{k-c}}\right]^{-1}$$
$$= \left[\sum_{k=0}^{c-1} \frac{(c\rho)^k}{k\,!} + \frac{(c\rho)^c}{c\,!}\ \frac{1}{1-\rho}\right]^{-1} \qquad (6)$$

Average of No. of Customers

- average number of customers $\bar{N}$:

$$\begin{aligned}\bar{N} &= \sum_{k=0}^{\infty} k p_k \\ &= \text{some cumbursome expressions}\end{aligned}$$

Variance of No. of Customers

- variance of the number of customers σ_N^2:

$$\begin{aligned}\sigma_N^2 &= \sum_{k=0}^{\infty}(k-\rho)^2 p_k \\ &= \text{some cumbursome expressions}\end{aligned}$$

Average Waiting Time

- Little's law can be used to compute T

Erlang's C formula

- associated with $M/M/c$ queues
- this computes a probability of wide use in telephony
- it is the probability that no server is available for an arriving customer in a system of c servers [4]
- usually denoted by $C(c, \lambda/\mu)$

[4] From a telephony point of view, trunks play the role of servers and calls play the role of customers.

Erlang's C formula

- thus, we have

$$
\begin{aligned}
C(c, \lambda/\mu) &\triangleq P[\text{no servers available for an arriving customer}] \\
&= P[\text{an arriving customer has a nonzero wait in queue}] \\
&= P[\text{an arriving customer is forced to join the queue}] \\
&= \textstyle\sum_{k=c}^{\infty} p_k \\
&= 1 - \textstyle\sum_{k=0}^{c-1} p_k = 1 - p_0 \sum_{k=0}^{c-1} \frac{(c\rho)^k}{k\ !}
\end{aligned}
$$

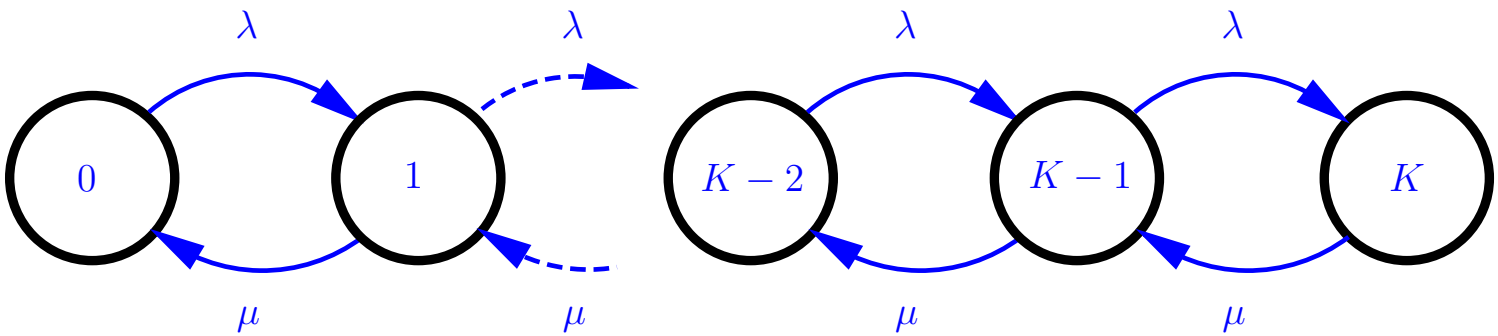
Erlang's C formula

- let us compute $C(c, \lambda/\mu)$

$$
\begin{aligned}
C(c, \lambda/\mu) &= 1 - p_0 \sum_{k=0}^{c-1} \frac{(c\rho)^k}{k\,!} \\
&\overset{(6)}{=} 1 - p_0 \left[\frac{1}{p_0} - \frac{(c\rho)^c}{c\,!} \frac{1}{1-\rho} \right] \\
&= p_0 \frac{(c\rho)^c}{c\,!} \frac{1}{1-\rho} \\
&= \frac{(c\rho)^c}{c\,!\,(1-\rho)} \left[\sum_{k=0}^{c-1} \frac{(c\rho)^k}{k\,!} + \frac{(c\rho)^c}{c\,!(1-\rho)} \right]^{-1} \\
&= \frac{(\lambda/\mu)^c}{c\,!\,(1-\rho)} \left[\sum_{k=0}^{c-1} \frac{(\lambda/\mu)^k}{k\,!} + \frac{(\lambda/\mu)^c}{c\,!(1-\rho)} \right]^{-1}
\end{aligned}
$$

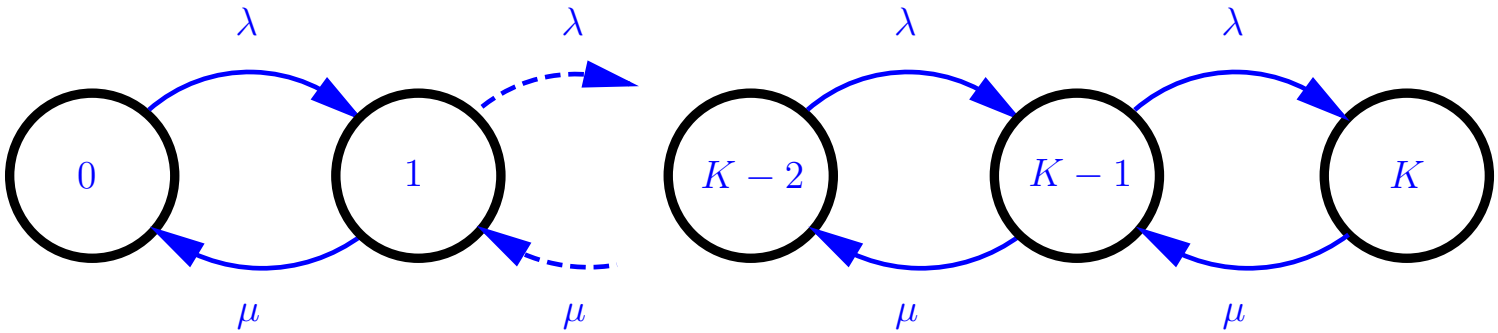
M/M/1/K, Finite Storage

M/M/1/K



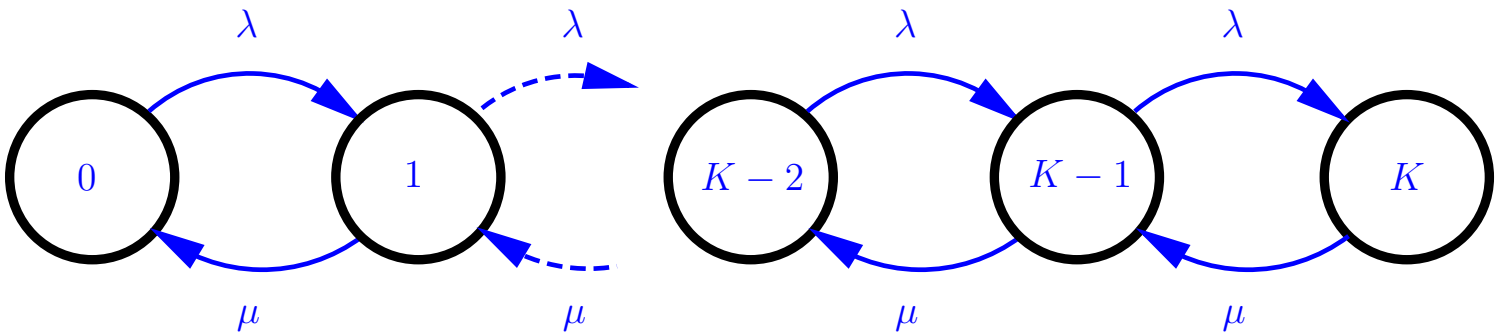
- there is one server
- system can accommodate only K customers

M/M/1/K



- any further arrivals depart immediately with no service (i.e., they are refused)
- refused customers are considered to be lost
- $K = 1 \rightarrow$ no waiting room $\rightarrow$ blocked calls cleared system with a single server

M/M/1/K



- here we have

$$\lambda_k = \begin{cases} \lambda & k < K \\ 0 & k \geq K \end{cases}$$

$$\mu_k = \mu \qquad\qquad k = 1, 2, \ldots, K$$

Existence of Equilibrium Solution

- system is always ergodic

Equilibrium Solution

- from (4), for $k \leq K$ we get

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{\mu}$$
$$= p_0 \left(\frac{\lambda}{\mu}\right)^k$$

- for $k > K$ we have

$$p_k = 0$$

Equilibrium Solution

- from (5), we get

$$
\begin{aligned}
p_0 &= \left[1 + \sum_{k=1}^{K} \left(\frac{\lambda}{\mu}\right)^k\right]^{-1} \\
&= \left[1 + \frac{(\lambda/\mu)(1-(\lambda/\mu)^K)}{1-\lambda/\mu}\right]^{-1} \\
&= \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{K+1}}
\end{aligned}
$$

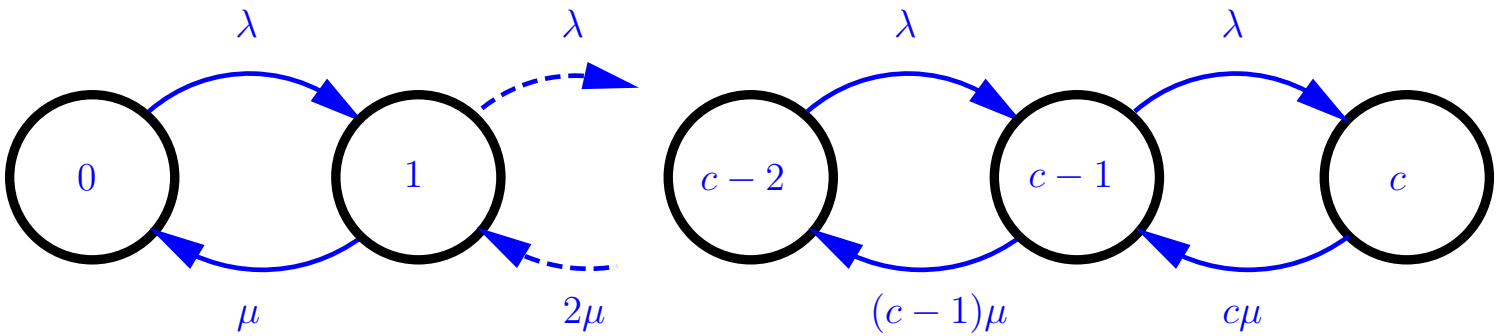
Equilibrium Solution for M/M/1/1

- here we have a blocked call cleared system with one server
- by substituting p_0 in p_k expressions, we get

$$p_k = \begin{cases} \frac{1}{1+\lambda/\mu} & k = 0 \\ \frac{\lambda/\mu}{1+\lambda/\mu} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

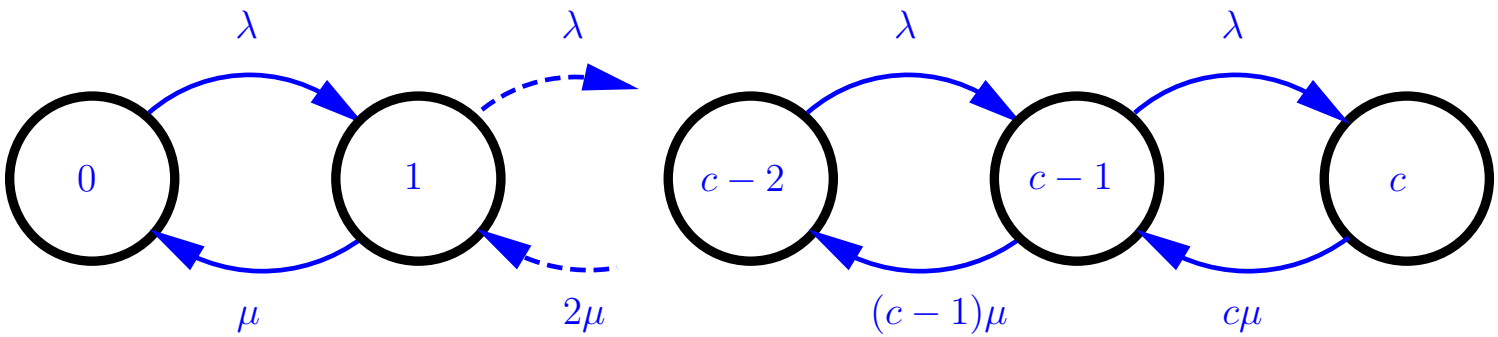
M/M/c/c, c-Sever Loss Systems

M/M/c/c



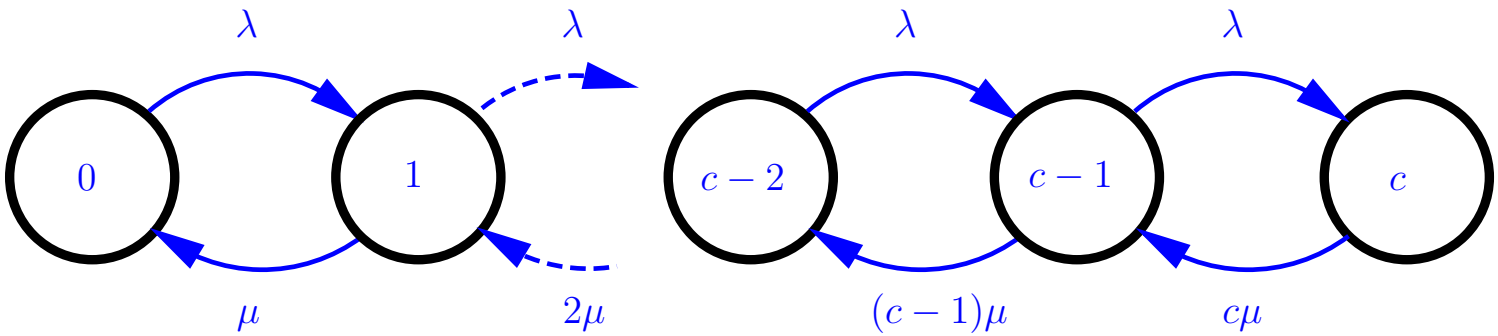
- a blocked call cleared system with c server
- system can accommodate only c customers
- any further arrivals depart immediately with no service (i.e., they are lost)

M/M/c/c



- this has been of great interest in the area of telephony

M/M/c/c



- here we have

$$\lambda_k = \begin{cases} \lambda & k < c \\ 0 & k \geq c \end{cases}$$

$$\mu_k = k\mu \qquad\qquad k = 1, 2, \ldots, c$$

Existence of Equilibrium Solution

- system is always ergodic

Equilibrium Solution

- from (4), for $k \leq c$ we get

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu}$$
$$= p_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k\,!}$$

- for $k > c$ we have

$$p_k = 0$$

Equilibrium Solution

- from (5), we get

$$p_0 = \left[\sum_{k=0}^{c} \left(\frac{\lambda}{\mu} \right)^k \frac{1}{k\,!} \right]^{-1}$$

Erlang's B formula

- note that p_c describes the fraction of time that all c servers are busy
- the formula for computing p_c is referred to as Erlang's B formula, i.e.,

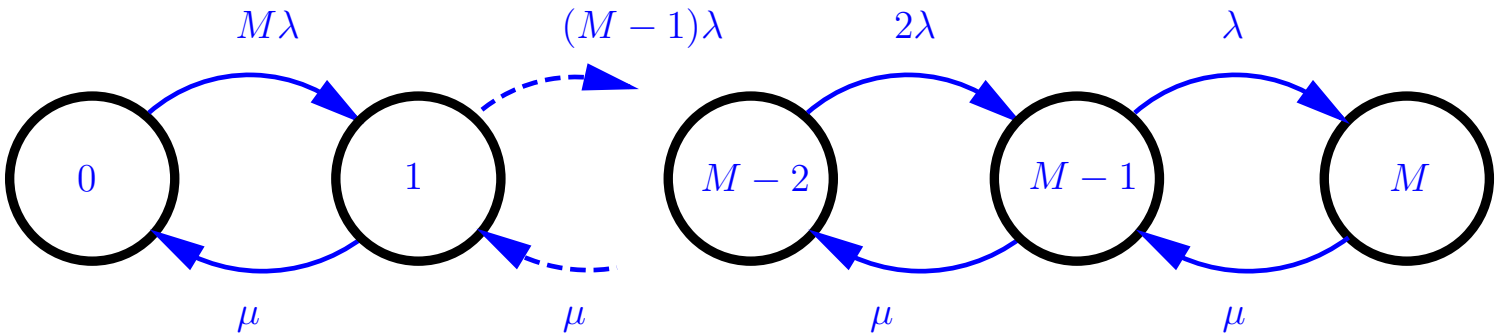
$$p_c = \frac{(\lambda/\mu)^c/(c\,!)}{\sum_{k=0}^{c}(\lambda/\mu)^k/(k\,!)} \triangleq B(c, \lambda/\mu) \tag{7}$$

M/M/1, with Finite M Customer Population

M/M/1 with Finite M Customers

- only one server
- no longer a Poisson input process with an infinite user population
- a finite population of possible users
- a customer is either
  - in the queue
  - at the server
  - outside, in some 'arriving'

M/M/1 with Finite M Customers



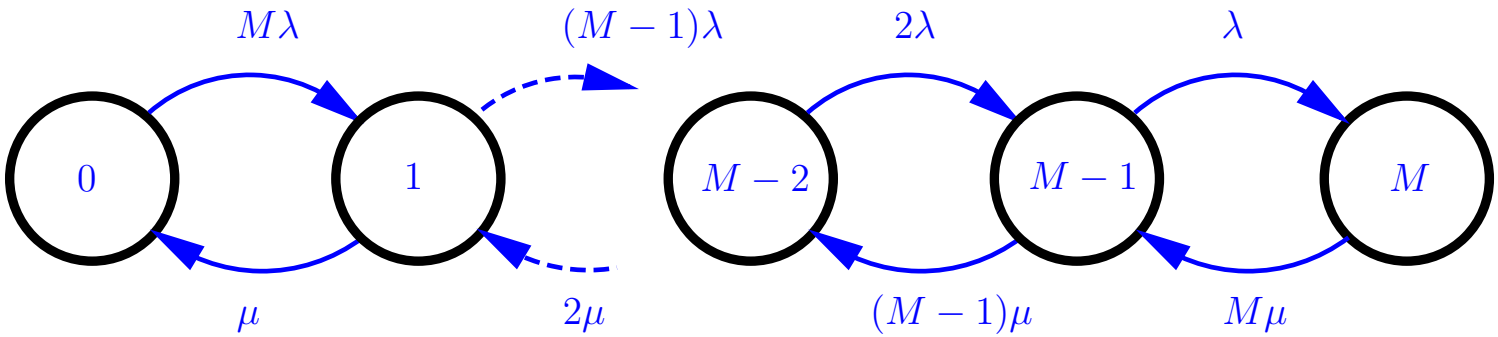
- the chain is ergodic
- equilibrium solutions exists
- can be computed by using the standard procedures we adopted earlier

M/M/∞, with Finite M Customer Population

M/M/∞ with Finite M Customers

- infinitely many servers
- no longer a Poisson input process with an infinite user population
- a finite population of possible users
- a customer is either
  - in the queue
  - at the server
  - outside, in some 'arriving'

M/M/∞ with Finite M Customers



- the chain is ergodic
- equilibrium solutions exists
- can be computed by using the standard procedures we adopted earlier

M/M/c/K, with Finite M Customer Population

M/M/c/K with Finite M Customers

- only c number of servers
- system can accommodate only K customers
- we have $c \leq K \leq M$
- no longer a Poisson input process with an infinite user population
- a finite population of possible users
- a customer is either in the queue, at the server, or outside, in some 'arriving'
- standard procedure applies for computing equilibrium solution