Traffic Engineering
EN3370

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Lecture 6: Elementary Queuing Theory I

Places to Read

- see [Kleinrock, § 3]

A Prelude

- reading assignment
  - § 1.2.7 of [Shortle]: discuss Kendall notation for queueing processes
  - § 1.4 up to § 1.4.1 (inclusive) of [Shortle]: Little's law
  - § 1.5 of [Shortle]: some general results

Birth-Death Queueing Systems in Equilibrium

Birth-Death Processes

- birth-death processes
  - a special case of a Markov process
  - transitions only to neighbouring states
  - the time between births are exponentially distributed
  - the time between deaths are each exponentially distributed

Basic Equations of Motion

- dynamics of the probability system

$$\frac{dP_k(t)}{dt} = -(\lambda_k + \mu_k)P_k(t) + \lambda_{k-1}P_{k-1}(t) + \mu_{k+1}P_{k+1}(t), \ k \geq 1$$
$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t), \ k = 0 \qquad (1)$$

- solution gives the transient behaviour of the queuing process

Basic Equations of Motion

- the solution become quickly unmanageable for sophisticated birth-death coefficients
- it is not always possible to solve for $P_k(t)$
- even if solved, the solution can be cumbersome $\rightarrow$ usefulness is questionable
- thus it is natural to check the behaviour when $t \rightarrow \infty$

Equilibrium Solution

Equilibrium Solution

- let us explore the limiting behaviour of these equations
- define [1]

$$p_k \triangleq \lim_{t \to \infty} P_k(t) \tag{2}$$

- p_k: the limiting probability that the system contains k members (or equivalently is in state k)
- let us assume the above limit exists and proceed [2]

[1] Here the notation is different to $\pi_k(t)$ and π_k used in lecture slides set 5, page 29, even though they carry the same meaning.

[2] See Appendix for necessary and sufficient conditions.

Equilibrium Solution

- from (1), as $t \to \infty$, we have

$$
\begin{aligned}
0 &= -(\lambda_k + \mu_k)p_k + \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}, \ k \geq 1 \\
0 &= -\lambda_0 p_0 + \mu_1 p_1, \ k = 0
\end{aligned} \tag{3}
$$

Equilibrium Solution

- we can concisely present (3) as

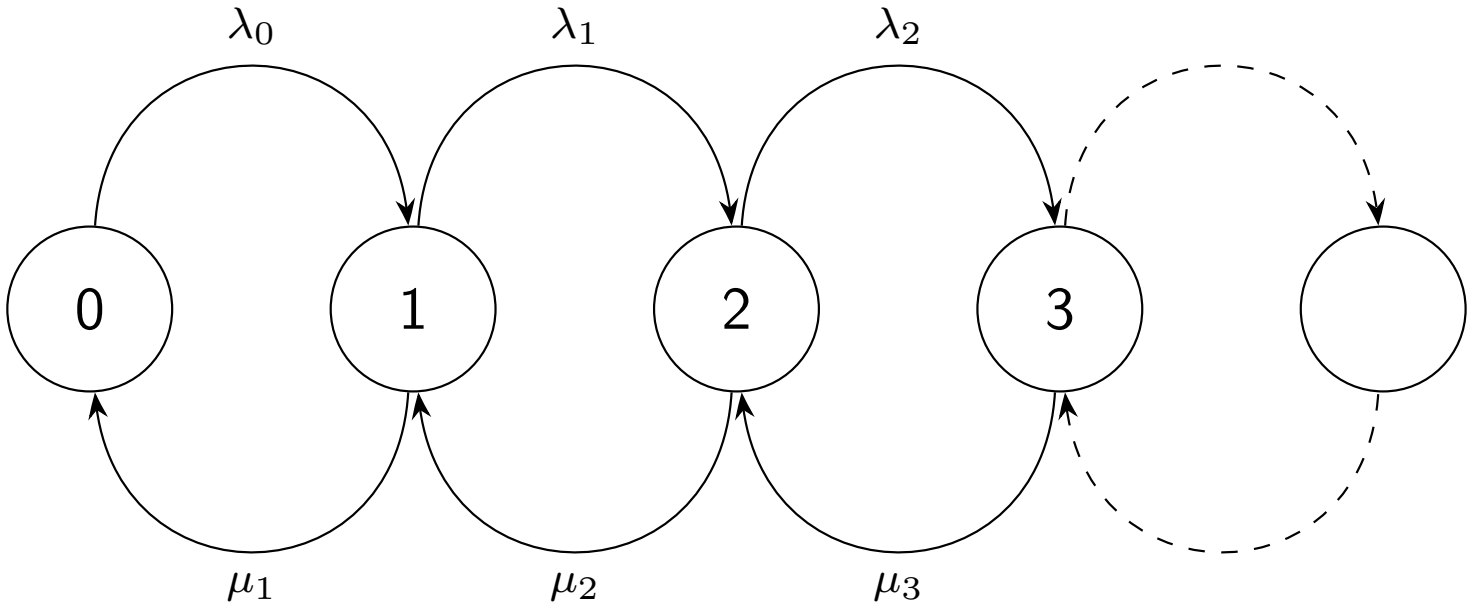
$$0 = -(\lambda_k + \mu_k)p_k + \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}, \ k \in \mathbb{Z} \qquad (4)$$

where we adopt the convention that $\lambda_{-1} = \lambda_{-2} = \cdots = 0$, $\mu_0 = \mu_{-1} = \mu_{-2} = \cdots = 0$, and $p_{-1} = p_{-2} = \cdots = 0$

- in addition we have the conservation relation:

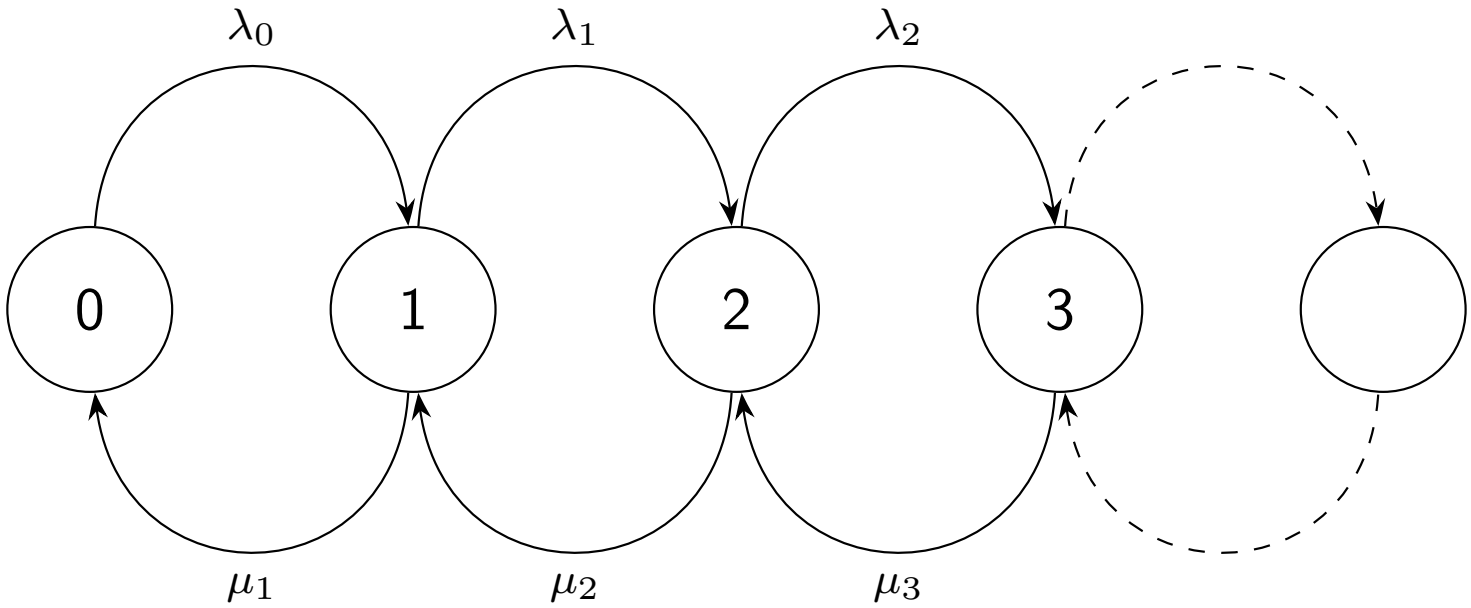
$$\sum_{k=0}^{\infty} p_k = 1 \qquad (5)$$

Equilibrium Solution



- equilibrium equations (4) directly from that diagram above
  - flow rate into state k = flow rate out of state k
  - e.g., state 2: $\lambda_1 p_1 + \mu_3 p_3 = (\lambda_2 + \mu_2) p_2$

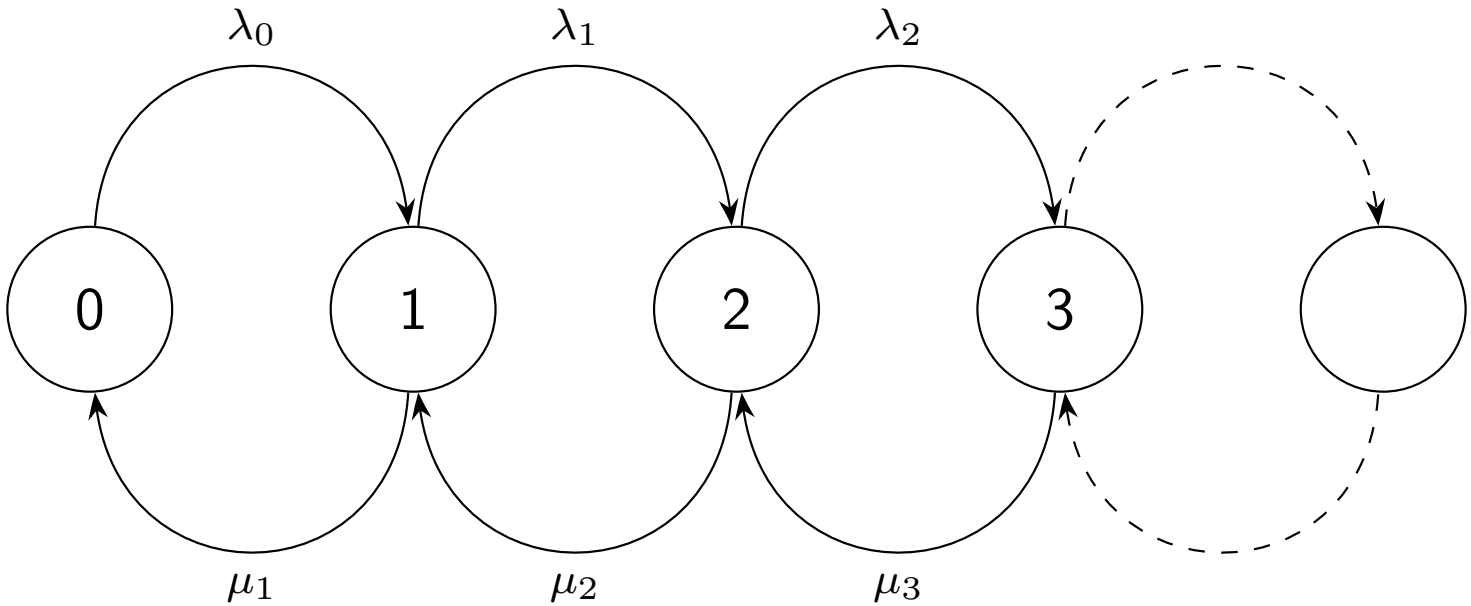
Equilibrium Solution



- in general we have

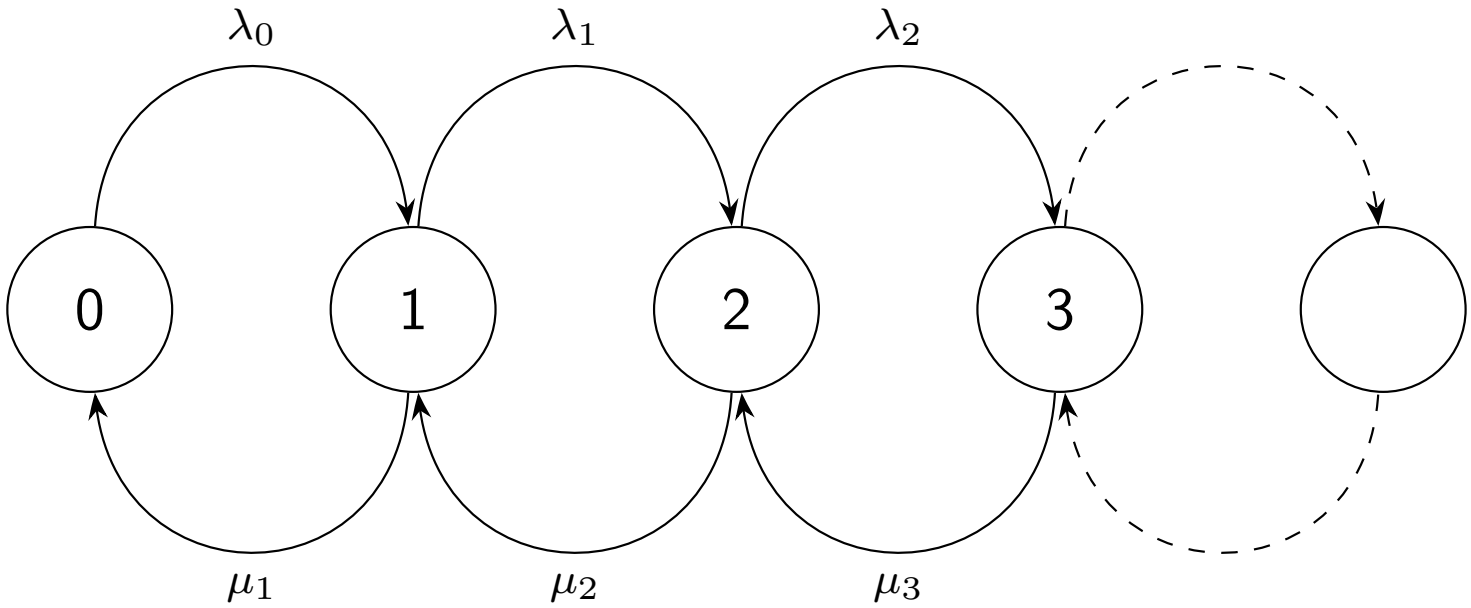
$$\lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1} = (\lambda_k + \mu_k)p_k \tag{6}$$

Equilibrium Solution



- equivalent equilibrium equations for (4)?
  - flow across the boundary between adjacent states
  - e.g., boundary between 1 and 2: $\lambda_1 p_1 = \mu_2 p_2$

Equilibrium Solution



- in general we have

$$\lambda_{k-1} p_{k-1} = \mu_k p_k \tag{7}$$

Computing the Equilibrium Solution

Equilibrium Solution

- both (6) and (7) can be used to get the equilibrium solution

Computing the Solution

- solve for p_1 in terms of p_0 by using (7)
- i.e.,

$$p_1 = \frac{\lambda_0}{\mu_1}\, p_0$$

- solve for p_2 in terms of p_1 by using (7)
- i.e.,

$$p_2 = \frac{\lambda_1}{\mu_2}\, p_1 = \frac{\lambda_1}{\mu_2}\, \frac{\lambda_0}{\mu_1}\, p_0 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2}\, p_0$$

- recursion yields

$$p_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k}\, p_0 = p_0 \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}, \quad k = 0, 1, 2, \ldots \quad (8)$$

Computing the Solution

- you may validate the solution (8) by substituting it in (6)
- conservation relation (5) can be used to compute p_0, i.e.,

$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}} \tag{9}$$

- the equations (9) and (8) are used in the rest of this lecture

Elementary Queues

M/M/1, The Classical Queuing System

M/M/1

- birth-death coefficients:

$$\lambda_k = \lambda \qquad k = 0, 1, 2, \ldots$$
$$\mu_k = \mu \qquad k = 0, 1, 2, \ldots$$

- average interarrival time $\bar{t} = 1/\lambda$
- average service time $\bar{x} = 1/\mu$

Existence of Equilibrium Solution

- necessary and sufficient conditions for the existence of equilibrium probabilities $\{p_k\}$: [3]

$$\lambda < \mu$$

[3] See the sums S_1 and S_2 in appendix.

Equilibrium Solution

- from (8), we get

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{\mu}$$
$$= p_0 \left(\frac{\lambda}{\mu}\right)^k , \qquad k \geq 0$$

- from (9), we get

$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^k} = 1 - \frac{\lambda}{\mu}$$

- let $\rho = \lambda/\mu$, note that $0 \leq \rho < 1$

Equilibrium Solution

- thus, we have

$$p_0 = 1 - \rho > 0$$

- moreover,

$$p_k = (1 - \rho)\rho^k \qquad k = 0, 1, 2, \ldots$$

Average of No. of Customers

- average number of customers $\bar{N}$:

$$\begin{aligned}
\bar{N} &= \sum_{k=0}^{\infty} k p_k \\
&= (1-\rho) \sum_{k=0}^{\infty} k \rho^k = (1-\rho)\rho \sum_{k=0}^{\infty} k \rho^{k-1} \\
&= (1-\rho)\rho \sum_{k=0}^{\infty} \frac{d}{d\rho}\, \rho^k = (1-\rho)\rho\, \frac{d}{d\rho} \left(\sum_{k=0}^{\infty} \rho^k \right) \\
&= (1-\rho)\rho\, \frac{d}{d\rho} \left(\frac{1}{1-\rho} \right) = \frac{\rho}{1-\rho}
\end{aligned}$$

Variance of No. of Customers

- variance of the number of customers σ_N^2:

$$\begin{aligned}\sigma_N^2 &= \sum_{k=0}^{\infty}(k-\bar{N})^2 p_k \\ &= \frac{\rho}{(1-\rho)^2}\end{aligned}$$

Average Waiting Time

- let T denote the average time spent in the system
- λ: average customer arrival rate
- apply Little's law:

$$\bar{N} = \lambda\, T$$

- thus we have

$$T = \frac{\rho}{1-\rho} \times \frac{1}{\lambda} = \frac{(\lambda/\mu)}{1-(\lambda/\mu)}\ \frac{1}{\lambda} = \frac{(1/\mu)}{1-\rho}$$

Average Waiting Time

- draw T Vs ρ
- as $\rho \rightarrow \infty$ system grows in an unbounded fashion
- this behaviour is experienced in almost every queuing system

Other Parameters

- average of number of customers N_q in the queue
- expected queue size of non-empty queues
- waiting time distribution
- will be explored in a viva session !

Appendix

Existence of the Solution

- define the following summations:

$$S_1 \triangleq \sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}$$

$$S_2 \triangleq \sum_{k=0}^{\infty} \left(\frac{1}{\lambda_k \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}} \right)$$

Existence of the Solution

- a classification:

$$\text{all the states are ergodic} \iff S_1 < \infty, S_2 = \infty$$

$$\text{all the states are recurrent null} \iff S_1 = \infty, S_2 = \infty$$

$$\text{all the states are transient} \iff S_1 = \infty, S_2 < \infty$$

Existence of the Solution

- ergodic case gives rise to the existence of equilibrium probabilities $\{p_k\}$
- a sufficient condition for ergodicity:

$$\exists\ k_0 \text{ such that } \forall\ k \geq k_0,\ \frac{\lambda_k}{\mu_k} < 1$$

- the above condition is true within the scope of this lecture