Traffic Engineering
EN3370

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Lecture 5: Continuous-Time Markov Chains

Places to Read

- see [Kleinrock, § 2.4]
- see [Ross-3, § 6.1-6.3]

Continuous-Time Markov Chains

Continuous-Time Markov Chains

- consider a stochastic process $X(t)$ where $t \in \mathbb{R}$
- t denotes the continuous-time
- $X(t)$ takes on a finite or countable number of possible values
  - those are denoted by set of positive integers $\{1, 2, \ldots\}$
- if $X(t) = i$, the process is said to be in state i at time t

Continuous-Time Markov Chains

- suppose we have

$$P\Big[X(t+s)=j \;\Big|\; X(s)=i, X(u)=x(u), 0\leq u<s\Big] = P\Big[X(t+s)=j \;\Big|\; X(s)=i\Big] \quad (1)$$

  for all steps $s, y \geq 0$ and for all possible values (states)

- such a stochastic process is known as a Continuous-Time Markov Chains

Continuous-Time Markov Chains

- the development of the theory for continuous time parallels that for discrete time
- recall.. time the process stay in the current state:
  - geometrically distributed in discrete-time Markov chains
  - exponential distributed in continuous-time Markov chains

Transition Probabilities

Transition Probability

- define time-dependent transition probability:

$$p_{ij}(s,t) \triangleq P\big[X(t) = j \;\big|\; X(s) = i\big], \qquad s \leq t \tag{2}$$

- compare the above with multi-step transition probability in the case of nonhomogeneous DTM Chains [1]

[1] See (21) in the Appendix, page (31).

Transition Probability

- follow the derivation (22) in page (31)
- by using similar arguments, we can derive Chapman-Kolmogorov equations for CTM chains

Chapman-Kolmogorov Equations

Transition Probability

- we have

$$
\begin{aligned}
p_{ij}(s,t) &= P\big[X(t)=j \;\big|\; X(s)=i\big] \qquad (3)\\
&= \sum_k P\big[X(t)=j, X(u)=k \;\big|\; X(s)=i\big]\\
&= \sum_k P\big[X(t)=j \;\big|\; X(u)=k, X(s)=i\big]\\
&\qquad\qquad \times P\big[X(u)=k \;\big|\; X(s)=i\big]\\
&= \sum_k P\big[X(t)=j \;\big|\; X(u)=k\big] P\big[X(u)=k \;\big|\; X(s)=i\big]\\
&= \sum_k p_{ik}(s,u) p_{kj}(u,t) \qquad (4)
\end{aligned}
$$

- (4) is called Chapman-Kolmogorov equations for CTM chains

Transition Probability Matrix

Transition Probability Matrix

- define

$$H(s,t) \triangleq \Big[p_{ij}(s,t)\Big]$$

- then (4) can be written in matrix form:

$$H(s,t) = H(s,u)H(u,t), \qquad s \le u \le t \tag{5}$$

- we require that $H(t,t) = I$

Transition Rate Matrix

Transition Rate Matrix $Q(t)$

- recall the 1-step transition probability matrix $P[n]$ for DTM chains [2], i.e.,

$$P[n] = H[n, n+1] = \Big[p_{ij}[n, n+1]\Big]$$

- for CTM chains the one-step transition probabilities are replaced by infinitesimal rates

- to this end let us define

$$P(t) \triangleq \Big[p_{ij}(t, t+\Delta t)\Big] = H(t, t+\Delta t)$$

[2] See the Appendix, pages 32-33.

Transition Rate Matrix $Q(t)$

- recall the 1-step transition probability matrix $P[n]$ for DTM chains [3], i.e.,

$$P[n] = H[n, n+1] = \Big[p_{ij}[n, n+1] \Big]$$

- for CTM chains the one-step transition probabilities are replaced by infinitesimal rates, which are the elements of $Q(t)$

[3] See the Appendix, pages 32-33.

Transition Rate Matrix $Q(t)$

- to find $Q(t)$, let us first define

$$P(t) \triangleq \left[p_{ij}(t, t+\Delta t)\right] = H(t, t+\Delta t)$$

Transition Rate Matrix $Q(t)$

- then consider the difference of $H(s, t + \Delta t)$ and $H(s, t)$: [4]

$$\begin{aligned} H(s, t + \Delta t) - H(s, t) &= H(s, t)H(t, t + \Delta t) - H(s, t) \\ &= H(s, t)\big(H(t, t + \Delta t) - I\big) \\ &= H(s, t)\big(P(t) - I\big) \end{aligned} \tag{6}$$

[4] See (5).

Transition Rate Matrix $Q(t)$

- next divide (6) by Δt and let $\Delta t \to 0$, i.e.,

$$\lim_{\Delta t \to 0} \frac{H(s, t+\Delta t) - H(s,t)}{\Delta t} = H(s,t) \lim_{\Delta t \to 0} \frac{P(t) - I}{\Delta t}$$

$$\frac{\partial H(s,t)}{\partial t} = H(s,t)Q(t) \tag{7}$$

Transition Rate Matrix $Q(t)$

- note the definition of $Q(t)$:

$$Q(t) \triangleq \lim_{\Delta t \to 0} \frac{P(t) - I}{\Delta t} \tag{8}$$

- from (8) we have

$$q_{ii}(t) \triangleq \lim_{\Delta t \to 0} \frac{p_{ii}(t, t + \Delta t) - 1}{\Delta t} \tag{9}$$

$$q_{ij}(t) \triangleq \lim_{\Delta t \to 0} \frac{p_{ij}(t, t + \Delta t)}{\Delta t}, \qquad i \neq j \tag{10}$$

- note that $\sum_j q_{ij}(t) = 0$ for all i

Interpret $Q(t)$

- diagonal elements, $q_{ii}(t)$

$$\begin{aligned} -q_{ii}(t)\Delta t + o(\Delta t) &= 1 - p_{ii}(t, t+\Delta t) \\ &= 1 - P\big[X(t+\Delta t) = i \;\big|\; X(t) = i\big] \end{aligned}$$

- given that the system is in state i at time t, the conditional probability that a transition occurs to any state other than i during the interval $(t, t+\Delta t)$

- $-q_{ii}(t) \rightarrow$ the rate at which the process departs from state i when it is in that state

Interpret $Q(t)$

- off-diagonal elements, $q_{ij}(t)$

$$\begin{aligned} q_{ij}(t)\Delta t + o(\Delta t) &= p_{ij}(t, t+\Delta t) \\ &= P\big[X(t+\Delta t) = j \;\big|\; X(t) = i\big] \end{aligned}$$

- given that the system is in state i at time t, the conditional probability that a transition occurs to state j during the interval $(t, t+\Delta t)$

- $q_{ij}(t) \rightarrow$ the rate at which the process departs from state i to state j when it is in state i

Forward and Backward Equations

Chap.-Kolmog. Equations, Matrix Form

- the forward Chapman-Kolmogorov equation for the CTM chains is [5]

$$\frac{\partial H(s,t)}{\partial t} = H(s,t)Q(t) \qquad s \leq t \tag{11}$$

- similarly, the backward Chapman-Kolmogorov equation for the CTM chains is

$$\frac{\partial H(s,t)}{\partial t} = -Q(s)H(s,t) \qquad s \leq t \tag{12}$$

[5] See (7).

Chap.-Kolmog. Equations, Matrix Form

- the solution to either (18) or (12) is

$$H(s,t) = \exp\left[\int_s^t Q(u)du\right] \tag{13}$$

Chap.-Kolmog. Equations, Scalar Form

- the forward Chapman-Kolmogorov equation for the CTM chains is

$$\frac{\partial p_{ij}(s,t)}{\partial t} = q_{jj}(t)p_{ij}(s,t) + \sum_{k \neq j} q_{kj}(t)p_{ik}(s,t) \qquad s \leq t$$

- similarly, the backward Chapman-Kolmogorov equation for the CTM chains is

$$\frac{\partial p_{ij}(s,t)}{\partial t} = -q_{ii}(s)p_{ij}(s,t) - \sum_{k \neq j} q_{ik}(s)p_{kj}(s,t) \qquad s \leq t$$

Transient Behaviour

Transient Behaviour

- recall for nonhomogeneous DTM chains [6]

$$\pi^{(n)} = \pi^{(0)} H[0, n] \tag{14}$$

- similarly for nonhomogeneous CTM chains we have

$$\pi(t) = \pi(0) H(0, t) \tag{15}$$

where

$$\pi(t) \triangleq [\pi_1(t)\ \pi_2(t)\ \pi_3(t)\ \cdots] \quad \text{and}$$

$$\pi_j(t) \triangleq P[X(t) = j]$$

[6] See (25), in the Appendix, page 37.

Transient Behaviour

- note from (20) that (15) is equivalent to

$$\pi(t) = \pi(0)\ \exp\left[\int_0^t Q(u)du\right]$$

- the matrix differential equation corresponding to above equation is

$$\frac{d\pi(t)}{dt} = \pi(t)Q(t)$$

- the scalar form is given as

$$\frac{d\pi_j(t)}{dt} = q_{jj}(t)\pi_j(t) + \sum_{k\neq j} q_{kj}(t)\pi_k(t) \qquad j = 1, 2, \ldots$$

Homogeneous CTM Chains

No Dependence upon Time

- in this case we drop the dependence upon time
- accordingly the following definitions are useful:

$$
\begin{aligned}
p_{ij}(t) &\triangleq p_{ij}(s, s+t) \\
q_{ij} &\triangleq q_{ij}(t) \\
H(t) &\triangleq H(s, s+t) = \left[p_{ij}(t)\right] \\
Q &\triangleq Q(t) = \left[q_{ij}\right]
\end{aligned}
$$

Chapman-Kolmogorov Equations

Chapman-Kolmogorov Equations

- Chapman-Kolmogorov equations become

$$p_{ij}(s+t) = \sum_k p_{ik}(s) p_{kj}(t) \tag{16}$$

- compare the above with (4)
- in matrix matrix form we have

$$H(s+t) = H(s)H(t) \tag{17}$$

- compare the above with (5)

Forward and Backward Equations

Forward and Backward Equations

- the forward Chapman-Kolmogorov equation for the homogeneous CTM chains is

$$\frac{dH(t)}{dt} = H(t)Q \tag{18}$$

- similarly, the backward Chapman-Kolmogorov equation for the homogeneous CTM chains is

$$\frac{dH(t)}{dt} = QH(t) \tag{19}$$

Forward and Backward Equations

- the solution to forward/backward equation is

$$H(t) = \exp\left[\, Qt \,\right] \tag{20}$$

Forward and Backward Equations

- the forward Chapman-Kolmogorov equation for the homogeneous CTM chains is

$$\frac{dp_{ij}(t)}{dt} = q_{jj}p_{ij}(t) + \sum_{k \neq j} q_{kj}p_{ik}(t)$$

- similarly, the backward Chapman-Kolmogorov equation for the homogeneous CTM chains is

$$\frac{dp_{ij}(t)}{dt} = q_{ii}p_{ij}(t) + \sum_{k \neq j} q_{ik}p_{kj}(t)$$

Transient Behaviour

Transient Behaviour

- the matrix differential equation for homogeneous CTM chains

$$\frac{d\pi(t)}{dt} = \pi(t)Q$$

- the scalar form is given as

$$\frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{k \neq j} q_{kj}\pi_k(t) \qquad j = 1, 2, \ldots$$

Transient Behaviour

- for an irreducible homogeneous Markov chain the following limit always exists:

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j$$

- the limit is independent of the initial state of the chain
- we further have

$$\lim_{t\to\infty} \pi_j(t) = \pi_j$$

- π_j forms the limiting state probability distribution

Limiting Distribution

- limiting distribution is given as the solution of [7]

$$q_{jj}\pi_j + \sum_{k \neq j} q_{kj}\pi_k = 0 \qquad \text{and} \qquad \sum_i \pi_i = 1$$

- the matrix form of the above is given by

$$\pi Q = 0 \qquad \text{and} \qquad 1^{\mathsf{T}}\pi = 1$$

[7] The forward equation can be used to derive this.

Appendix
Nonhomogeneous DTM Chains

Multi-Step Transition Probability

- multistep transition probabilities: define

$$
\begin{aligned}
p_{ij}[m,n] &\triangleq P\big[X_n = j \;\big|\; X_m = i\big] && (21)\\
&= \sum_k P\big[X_n = j, X_q = k \;\big|\; X_m = i\big] \\
&= \sum_k P\big[X_n = j \;\big|\; X_q = k, X_m = i\big] P\big[X_q = k \;\big|\; X_m = i\big] \\
&= \sum_k P\big[X_n = j \;\big|\; X_q = k\big] P\big[X_q = k \;\big|\; X_m = i\big] \\
&= \sum_k p_{ik}[m,q] p_{kj}[q,n] && (22)
\end{aligned}
$$

- (22) is called Chapman-Kolmogorov equations for DTM chains

1-Step Transition Probability Matrix

- 1-step transition probability matrix $P[n]$ at time n
- define

$$P[n] \triangleq \Big[p_{ij}[n, n+1]\Big]$$

- in the case of homogeneous DTM chains
  - $P[n] = P$ for all n
  - n-step transition probability matrix was P^n
- however, for nonhomogeneous DTM chains, the above do not hold

Multi-Step Transition Probability Matrix

- define

$$H[m,n] \triangleq \Big[p_{ij}[m,n]\Big]$$

- note: $H[n,n+1] = P[n]$
- (22) can be written in matrix form:

$$H[m,n] = H[m,q]H[q,n], \qquad m \leq q \leq n$$

- we require that $H[n,n] = I$

Express $H[m, n]$ in terms of $P[n]$

Forward Chapman-Kolmog. Equations

- let $q = n - 1$ in (22) above

$$p_{ij}[m, n] = \sum_k p_{ik}[m, n-1] p_{kj}[n-1, n]$$

- matrix form is simply

$$\begin{aligned} H[m, n] &= H[m, n-1]) H[n-1, n] \\ &= H[m, n-1] P[n-1] \end{aligned} \tag{23}$$

Backward Chapman-Kolmog. Equations

- let $q = m + 1$ in (22) above

$$p_{ij}[m, n] = \sum_k p_{ik}[m, m + 1]p_{kj}[m + 1, n]$$

- matrix form is simply

$$\begin{aligned} H[m, n] &= H[m, m + 1]H[m + 1, n] \\ &= P[m]H[m + 1, n] \end{aligned} \tag{24}$$

Express $H[m,n]$ in terms of $P[n]$

- (23) or (24) $\rightarrow$ for nonhomogeneous DTM chains, we get

$$H[m,n] = \underbrace{P[m]P[m+1]\cdots P[n-1]}_{n-1 \text{ martrices here}}, \qquad m \leq n-1$$

- recall for homogeneous DTM chains, we had

$$H[m,n] = (n-m)\text{-step probability} = P^{n-m}$$

Transient Behaviour

Transient Behaviour

- recall for homogeneous DTM chains

$$\pi^{(n)} = \pi^{(n-1)} P$$

- similarly for nonhomogeneous DTM chains we can obtain

$$\pi^{(n)} = \pi^{(n-1)} P[n-1]$$

- recursion yields

$$\pi^{(n)} = \pi^{(0)} \underbrace{P[0]P[1]\cdots P[n-1])}_{H[0,n]} \tag{25}$$