Traffic Engineering
EN3370

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Lecture 4: Discrete-Time Markov Chains

Places to Read

- see [Kleinrock, § 2.2-2.3]
- see [Ross-3, § 4.1-4.3]

A Classification of Random Processes

Classification Depends on ?

- the state space
- the index (time) parameter
- statistical dependencies among the random variables $X(t)$ for different values of the index parameter t

Classify Based on the State Space

The State Space

- consider the set of possible values (or states) that $X(t)$ may take on
- in other words the positions that particle may occupy
- suppose the set of possible values (e.g., positions) are
  - finite or countable
- then we say the random process a discrete-state process or a chain

The State Space

- suppose the set of possible values (e.g., positions) are over a
  - (finite or infinite) continuous interval
  - or set of such intervals
- then we say the random process a continuous-state process

Classify Based on the Index (Time) Parameter

The Index (Time) Parameter

- suppose the permitted times at which changes in position may take place are
  - finite or countable
- then we say the random process a discrete-time parameter process

The Index (Time) Parameter

- suppose the permitted times at which changes in position may take place are
  - anywhere within (a set of) finite or infinite intervals on the time axis
- then we say the random process a continuous-time parameter process

Classify Based on Statistical Dependencies

Statistical Dependencies

- complete distribution function among the random variable (vector) $X = [X(t_1), X(t_2), \ldots, X(t_n)]^{\mathsf{T}}$ should be available
- i.e., one must specify

$$F_X(x;t) \triangleq P[X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n]$$

for all $x = [x_1, x_2, \ldots, x_n]^{\mathsf{T}}$, $t = [t_1, t_2, \ldots, t_n]^{\mathsf{T}}$, and n

- in fact having the above is a formidable task

Statistical Dependencies [1]

- stationary processes
- independent processes
- Markov processes
- birth-death processes
- semi-Markov processes
- random walks
- renewal process

[1] See § 2.2 of [Kleinrock]

Markov Chains

Markov Chains

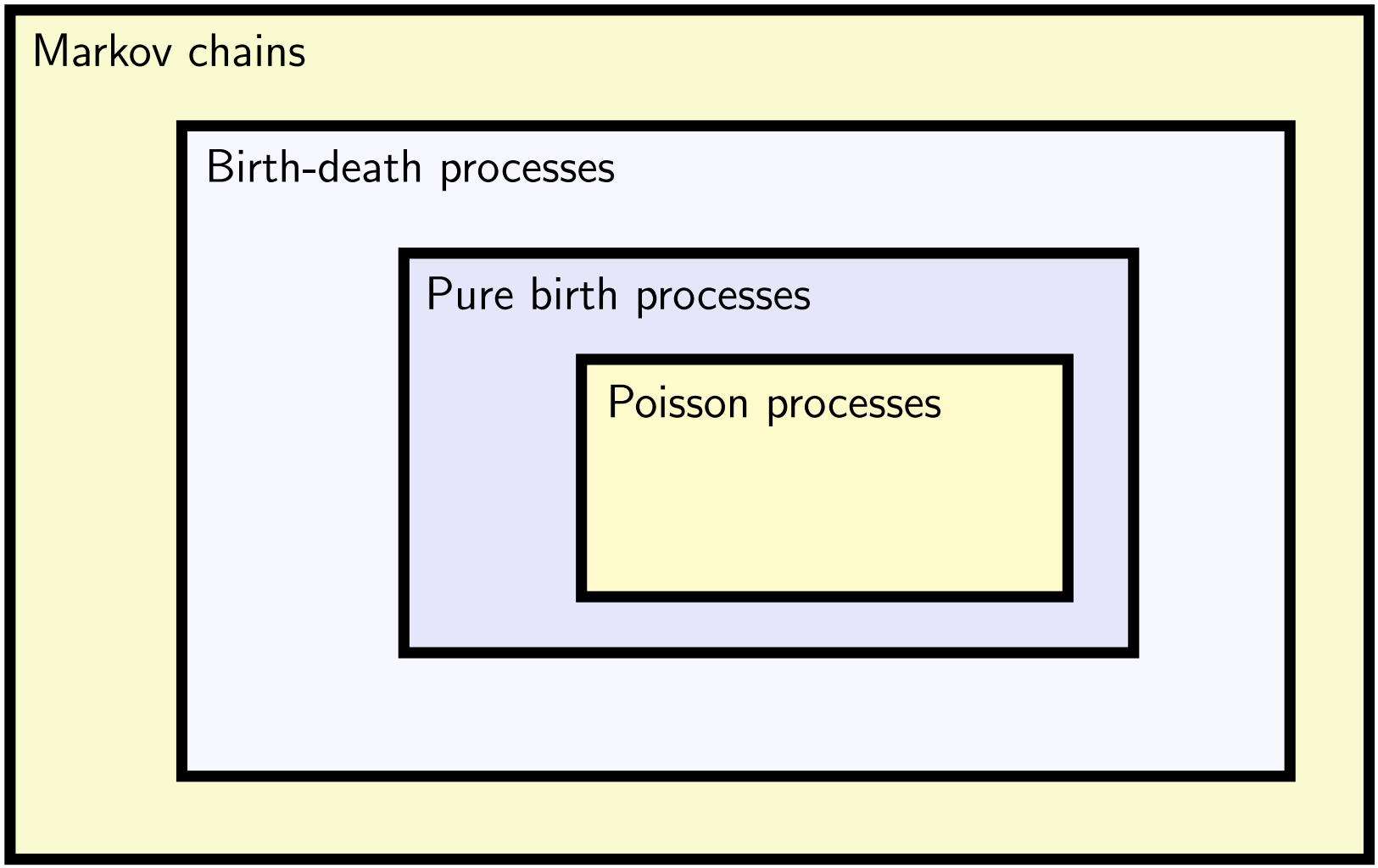
- Markov processes with discrete state space $\rightarrow$ Markov chains
- Markov chains
  - discrete-time Markov chains
  - continuous-time Markov chains
- time the process stay in the current state:
  - geometrically distributed in discrete-time Markov chains
  - exponential distributed in continuous-time Markov chains

Birth-Death Processes

Birth-Death Processes

- a particular class of Markov chains
- birth-death processes
  - discrete-time birth death processes
  - continuous-time birth death processes
- state transitions are between neighbouring states only

A Basic Relationship [2]



[2] See Figure 2.4 of [Kleinrock] for a more general view

Discrete-Time Markov Chains

Discrete-Time Markov Chains

- consider a stochastic process X_n where $n = 0, 1, 2, \ldots$
- n denotes the discrete-time (or steps)
- X_n takes on a finite or countable number of possible values
  - those are denoted by set of positive integers $\{1, 2, \ldots\}$
- if $X_n = i$, the process is said to be in state i at time (step) n

Discrete-Time Markov Chains

- suppose we have

$$P\left[X_{n+1}=j \,\middle|\, X_n=i, X_{n-1}=i_{n-1}, \ldots, X_1=i_1, X_0=i_0\right] = P\left[X_{n+1}=j \,\middle|\, X_n=i\right]$$

  for all steps n and for all possible values (states)

- such a stochastic process is known as a Discrete-Time Markov Chains

- RHS of above $\rightarrow$ the (one-step) transition probability

Transition Probabilities

One-Step Transition Probability

- (one-step) transition probability: $P[X_{n+1} = j \mid X_n = i]$
- the conditional probability of making a transition from state i at step n to state j at the $(n+1)$th step in the process

Homogeneous Markov Chain

- suppose the transition probabilities are independent of n
- such a Markov chain is referred to as a homogeneous Markov chain
- thus, we define, for all n

$$p_{ij} \triangleq P\big[X_{n+1} = j \;\big|\; X_n = i\big]$$

- we consider homogeneous Markov chains only, unless otherwise stated

m-Step Transition Probability

- m-step transition probability: $p_{ij}^{(m)}$
- we have

$$p_{ij}^{(m)} \triangleq P\left[X_{n+m} = j \mid X_n = i\right] \tag{1}$$

$$= \sum_k p_{ik}^{(m-1)} p_{kj} \tag{2}$$

$$= \sum_k p_{ik}^{(m-n)} p_{kj}^{(n)} \quad \forall\, m, \forall n \in \{1, 2, \ldots, (m-1)\} \tag{3}$$

m-Step Transition Probability

- let $P^{(m)}$ denote the matrix of m-step transition probabilities
- then, (1)-(3) simply yield

$$P^{(m)} = P^m$$

where P is the transition probability matrix, i.e., $[P]_{i,j} = p_{ij}$

Irreducible Markov Chains

Irreducibility

- state j is said to be accessible from state i if $p_{ij}^{(n)} > 0$ for some integer n
- two states i and j that are accessible to each other are said to communicate
- Markov chain is irreducible: every state can be reached (accessible) from every other state, i.e.,
  - $\forall$ pair of states (i, j), $\exists$ an integer n such that $p_{ij}^{(n)} > 0$
  - or all states communicate with each other

Classification of States

Periodic and Aperiodic States

- m-step transition probability: $p_{ii}^{(n)}$ is used
- compute the set of steps such that $p_{ii}^{(n)} > 0$
- i.e., the set $\left\{ n \mid p_{ii}^{(n)} > 0 \right\}$
- the period of state i is the greatest common divisor γ_i of the integer elements in the set above
- $\gamma_i = 1 \rightarrow$ state i is aperiodic
- $\gamma_i > 1 \rightarrow$ state i is periodic with period γ_i

Recurrent and Transient States

- it is useful to define a key probability $f_j^{(n)}$
- first return to state j in n steps after leaving j

$$f_j^{(n)} \triangleq P[\text{ first return to } j \text{ occurs } n \text{ steps after leaving } j \text{ }]$$

- ever returning to state j after leaving j

$$f_j \triangleq P[\text{ever returning to } j \text{ after leaving } j] = \sum_{n=1}^{\infty} f_j^{(n)}$$

- $f_j = 1 \rightarrow$ state j is recurrent
- $f_j < 1 \rightarrow$ state j is transient

Further Classification of Recurrent States

- recall.. $f_j = 1 \rightarrow$ state j is recurrent
- define the mean recurrence time M_j:

$$M_j \triangleq \sum_{n=1}^{\infty} n f_j^{(n)}$$

- $M_j = \infty \rightarrow$ state j is recurrent null or null recurrent
- $M_j < \infty \rightarrow$ state j is recurrent nonnull or positive recurrent

Ergodic States

- a state i is said to be ergodic:
  - if state i aperiodic and recurrent nonnull
  - i.e., if $\gamma_i = 1$, $f_i = 1$, and $M_i < \infty$
- all states are ergodic $\rightarrow$ ergodic Markov chain

States of an Irreducible Markov Chain

States of an Irreducible Markov Chain

Theorem (page 29, of Kleinrock)

The states of an irreducible Markov chain are either all transient or all recurrent nonnull or all recurrent null. If periodic, then all states have the same period γ.

State Probabilities

State Probabilities

- define the probability of finding the system in state j at the nth step, i.e.,

$$\pi_j^{(n)} \triangleq P[X_n = j]$$

Stationary Distribution

- suppose we choose $\pi_j^{(0)} = p_j$ for all n, where p_j is a probability distribution
- i.e., p_j is the initial probability distribution that we have chosen
- then, suppose that for all n we have $\pi_j^{(n)} = p_j$
- such a probability distribution p_j is said to be a stationary distribution

Limiting Probabilities

Theorem (page 29, of Kleinrock)

In an irreducible and aperiodic Markov chain the limiting probabilities $\pi_j = \lim_{n\to\infty} \pi_j^{(n)}$ *always exist and are independent of the initial probability distribution.*

Moreover, either

1. *all states are transient or all states are recurrent null in which cases* $\pi_j = 0$ *for all* j *and there exists no stationary distribution,*

   or

2. *all states are recurrent nonnull and then* $\pi_j > 0$ *for all* j*, in which case the set* π_j *is a stationary probability distribution and* $\pi_j = (1/M_j)$*. In this case* π_j *are uniquely determined by using*

$$\sum_i \pi_i = 1, \qquad \pi_j = \sum_i \pi_i p_{ij}$$



Transient Behaviour of the System

Transient Behaviour

- recall.. P is the transition probability matrix, i.e., $[P]_{i,j} = p_{ij}$
- useful to determine transient behaviour of the system: recall..

$$\pi_j^{(n)} \triangleq P[X_n = j]$$

- define the probability vector at step n as

$$\pi^{(n)} \triangleq \left[\pi_1^{(n)} \ \pi_2^{(n)} \ \pi_3^{(n)} \ \cdots\right]$$

Transient Behaviour

- thus, we can link $\pi^{(1)}$, $\pi^{(0)}$ and P as follows:

$$\pi^{(1)} = \pi^{(0)} P$$

- similarly, we can link $\pi^{(2)}$, $\pi^{(1)}$ and P as follows:

$$\pi^{(2)} = \pi^{(1)} P = \pi^{(0)} P^2$$

- recursion yields

$$\pi^{(n)} = \pi^{(n-1)} P = \pi^{(0)} P^n, \quad n = 1, 2, \ldots \tag{4}$$

Transient Behaviour

- assume the limiting probabilities $\pi_j = \lim_{n\to\infty} \pi_j^{(n)}$ always exist [3]
- thus from (5) we have

$$\lim_{n\to\infty} \pi^{(n)} = \lim_{n\to\infty} \pi^{(n-1)} P \tag{5}$$

- i.e.,

$$\pi = \pi P \tag{6}$$

where $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \cdots \end{bmatrix}$

[3] For example, consider an irreducible ergodic Markov chain.

Irreducible Ergodic Markov Chain

- recall the theorem in page 27 and the definition of ergodic Markov chains in page 23 above
- thus for ergodic Markov chains:
  - the limiting probabilities $\pi_j = \lim_{n\to\infty} \pi_j^{(n)}$ always exist
  - $\pi_j > 0$ for all j, in which case the set π_j is a stationary probability distribution and $\pi_j = (1/M_j)$. Moreover, π_j are uniquely determined by using

$$\underbrace{\sum_i \pi_i = 1,}_{1^{\mathrm{T}}\pi=1} \qquad \underbrace{\pi_j = \sum_i \pi_i p_{ij}}_{\pi=\pi P}$$

Irreducible Ergodic Markov Chain

- a Markov chain is said to be ergodic if the probability distribution $\pi_j^{(n)}$ as a function of n always converges to a limiting stationary distribution π_j, which is independent of the initial state distribution, [compare with (5) and (6)]

- it can be shown that all states of a finite irreducible aperiodic Markov chain are ergodic