Traffic Engineering
EN3370

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Lecture 3: The Poisson Processes

Places to Read

- content of this discussion is mainly from [Ross-3, § 5.3]
- of course you are welcome to check the related material from other course references as well, e.g., [Ross-1, § 2]

Counting Process

- a stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of events that occur by time t
- e.g.,
  - $N(t) =$ the number of persons that enters a shop at or prior to time t
  - in above example, event corresponds to a person entering the shop
- what if $N(t) =$ the number of persons in the shop at time t ?

Counting Process

- a counting process must satisfy the following:

  1) $N(t) \geq 0$

  2) $N(t)$ is integer valued

  3) $s < t \implies N(s) \leq N(t)$

  4) for $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$

Independent Increments

- a counting process is said to possess independent increments if the numbers of events that occur in disjoint time intervals are independent
- for example,
  - the number of events that occur by $t = 10$ [i.e., $N(10)$] must be independent of the number of events that occur over the range $10 < t \leq 15$ [i.e., $N(15) - N(10)$]

Stationary Increments

- a counting process is said to possess stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval
- in other words,
  - for all s, number of events in the interval $(s, s+t]$ has the same distribution
- now we are ready to discuss Poisson process!

Poisson Process (Definition 1)

- Poisson process is one of the most important counting processes
- the counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

  1) $N(0) = 0$
  2) the process has independent increments
  3) the number of events in any interval of length t is Poisson distributed with mean λt
     - i.e., for all $s, t \geq 0$

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \cdots$$

Poisson Process

- verify that the Poisson process has stationary increments
- show that $\mathbb{E}\{N(t)\} = \lambda t$
- to determine if an arbitrary counting process is actually a Poisson process, we must show that conditions (1), (2), and (3) above are satisfied
- usually it is not clear how to verify whether (3) is true!
- so some equivalent definitions

Poisson Process (Definition 2)

- the counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

  1) $N(0) = 0$

  2) the process has stationary and independent increments

  3) $P\{N(h) = 1\} = \lambda h + o(h)$

  4) $P\{N(h) \geq 2\} = o(h)$

- $o(\cdot)$ stands for little-oh notation , read Definition 5.2 and Example 5.12 of Ross-3

Poisson Process

- Definition 1 and Definition 2 are equivalent, i.e.,

$$\text{Definition 1} \iff \text{Definition 2}$$

- Definition 1 $\implies$ Definition 2: easily shown
- Definition 2 $\implies$ Definition 1: not trivial [1]

[1] See [pp. 61-62, Ross-1] or [pp. 305-306, Ross-3]

Poisson Process (Definition 3)

- if part (3) and part (4) of Definition 2 is modified so that they are true for any interval of length h, then we can give another equivalent definition, together with some little changes
  part (2)

- i.e., we can have another definition

Poisson Process (Definition 3)

- the counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

  1) $N(0) = 0$

  2) the process has independent increments

  3) $P\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$

  4) $P\{N(t+h) - N(t) \geq 2\} = o(h)$

Poisson Process

- Part (2) above:
  - knowing how many events in one time interval provides no information about how many events in another (non-overlapping) time interval

Poisson Process

- Part (3) above:
  - probability of observing exactly one event over a short time interval is approximately proportional to the length of the interval

Poisson Process

- property (4) above:
  - probability of observing two or more events in a short time interval is essentially zero
  - impossible for two events to occur exactly at the same time
  - called orderliness

Recall..Poisson Distribution

- a Poisson random RV is a discrete RV with probability mass function (pmf)

$$p_n = \frac{A^n}{n!} e^{-A}, \qquad n = 0, 1, \ldots$$

  where $A > 0$ is a constant

- when X is a Poisson RV, then
  - $\mathbb{E}\{X\} = A$
  - $\mathtt{Var}\{X\} = A$

Poisson Process

- let $\{N(t), t \geq 0\}$ be a Poisson process having rate λ, $\lambda > 0$
- the number of events occurring by time t is a Poisson RV with mean λt, i.e.,

$$P\{N(t) = n\} = p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \qquad n = 0, 1, \ldots$$

- recall the mean of the above Poisson RV is λt
- that is during $(0, t]$, we have on average λt events (e.g., call originations)
- thus, λ is called mean number of call arrivals in unit time

Interarrival Time Distribution

- consider a Poisson process
- X_1 : time of the first event
- X_n : time between $(n-1)$st and nth event
- $\{X_n, n = 1, 2, \ldots\}$: the sequence of interarrival times

Interarrival Time Distribution

- $X_n,\ n = 1, 2, \ldots$ are independently and identically distributed exponential random variables having mean λ
- read the proof from the reference

Waiting Time Distribution

- denote by S_n the arrival time of the nth event, i.e.,

$$S_n = \sum_{i=1}^{n} X_i, \qquad n = 1, 2, \ldots$$

- S_n is the waiting time until the nth event
- what is the distribution of S_n?

Waiting Time Distribution

- denote by S_n the arrival time of the nth event, i.e.,

$$S_n = \sum_{i=1}^{n} X_i, \qquad n = 1, 2, \ldots$$

- S_n is the waiting time until the nth event
- what is the distribution of S_n? gamma distribution with parameters n and λ
- i.e.,

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Interarrival and Waiting Time Distribution

- what is the probability that no calls originate in $(0, t]$?

Interarrival and Waiting Time Distribution

- what is the probability that no calls originate in $(0, t]$?

$$p_0(t) = e^{-\lambda t}$$

- what is the probability that the interarrival time X is no greater than t?

Interarrival and Waiting Time Distribution

- what is the probability that no calls originate in $(0, t]$?

$$p_0(t) = e^{-\lambda t}$$

- what is the probability that the interarrival time X is no greater than t?

$$F(t) = 1 - e^{-\lambda t}, \qquad t \geq 0$$

- try example 5.13 of [Ref-3]

Further Properties of Poisson Processes

Splitting of Poisson Processes

- consider a Poisson process $\{N(t), t \geq 0\}$ having rate λ
- events are classified as type I with probability p or as type II with probability $1 - p$, independently of all other events
- let $N_1(t)$ and $N_2(t)$ denote respectively the number of type I and type II events occurring in $[0, t]$, and thus we have $N(t) = N_1(t) + N_2(t)$

Splitting of Poisson Processes

- then we have the following result (see Proposition 5.2, [Ross-3])
- $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$ and the two processes are independent
- the result holds true for even when there are more than two types of event classification [2]

[2] see Proposition 5.3 [Ross-3] for a genetral proof.

Superposition of Poisson Processes

- consider 2 independent Poisson processes $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ having rate λ_1, and λ_1, respectively
- then we have the following result (see Proposition 5.4, [Ross-3]) [3]
- $\{N(t), t \geq 0\}$ is a Poisson processes having rate $\lambda = \lambda_1 + \lambda_2$
- the result holds true for even when there are more than one independent Poisson processes

[3] the proposition holds even under more general settings.

Conditional Distribution of Arrival Times

- consider a Poisson processes $\{N(t), t \geq 0\}$ having rate λ
- given that exactly one event has taken place by time t (i.e., a condition)
- what is the conditional distribution of the time X_1 (recall that $X_1 = S_1$) at which the first arrival occurred?
- in fact, under the condition above, the time of the event has the same distribution as a random variables uniformly distributed on the interval $[0, t]$.

Conditional Distribution of Arrival Times

- proof! first suppose $s < t$ (what if $s > t$?)

$$
\begin{aligned}
P\{X_1 < s \mid N(t) = 1\} &= \frac{P\{X_1 < s, N(t)\}}{P\{N(t) = 1\}} \\
&= \frac{P\{\text{one event over}(0, s], \text{no events over (s,t]}\}}{P\{N(t) = 1\}} \\
&= \frac{P\{N(s) = 1, N(t-s) = 0\}}{P\{N(t) = 1\}} \\
&= \frac{P\{N(s) = 1\} P\{N(t-s) = 0\}}{P\{N(t) = 1\}} \\
&= \frac{(\lambda s)^1 e^{-\lambda s}, (\lambda(t-s))^0 \, e^{-\lambda(t-s)}}{(\lambda t)^1 e^{-\lambda t}} \\
&= \frac{s}{t} \qquad \text{compare with a CDF of a uniform RV}
\end{aligned}
$$

Conditional Distribution of Arrival Times

- proof! another method, let $f(s \mid N(t) = 1)$, $(s < t)$ be the conditional density of the time X_1 at which the event occurred

$$
\begin{aligned}
f(s \mid N(t) = 1)\epsilon &\approx P\{X_1 \in [s, s+\epsilon] \mid N(t) = 1\} \\
&= \frac{P\{\text{one event over } [0,\epsilon], N(t) = 1\}}{P\{N(t) = 1\}} \\
&= \frac{P\{\text{one event over } [0,\epsilon], \text{no events over} (\epsilon, t]\}}{P\{N(t) = 1\}} \\
&= \frac{(\lambda\epsilon)^1 e^{-\lambda\epsilon}, (\lambda(t-\epsilon))^0 e^{-\lambda(t-\epsilon)}}{(\lambda t)^1 e^{-\lambda t}} \\
&= \frac{\epsilon}{t}
\end{aligned}
$$

- thus we have $f(s \mid N(t) = 1) = 1/t$ (compare with a pdf of a uniform RV)

Conditional Distribution of Arrival Times

- consider a Poisson processes $\{N(t), t \geq 0\}$ having rate λ
- given that exactly n events have taken place by time t (i.e., a condition)
- what is the conditional distribution of the times, $S_1, \ldots, S_n$ at which the first n arrivals occurred?
- in fact, under the condition above, the time of the arrivals have the same distribution as the order statistics [4] corresponding to n independent random variables uniformly distributed on the interval $[0, t]$

[4] See p. 317 of [Ros-3] or p. 276, § 6.6 [Ros-2].

Conditional Distribution of Arrival Times

- proof!, let $f(\boldsymbol{s} \mid N(t) = n)$, $(s_1 < s_2 < \ldots < s_n < t)$ be the conditional density of the time $S_1, \ldots, S_n$ at which the arrivals occurred $[\boldsymbol{s} = (s_1, \cdots, s_n)]$

$$
\begin{aligned}
f(\boldsymbol{s} \mid N(t) = 1)\epsilon^n &\approx P\{S_i \in [s_i, s_i + \epsilon], i = 1, \ldots, n \mid N(t) = n\} \\
&= \frac{P\{1 \text{ event over } [s_i, s_i + \epsilon], i = 1, \ldots, n, N(t) = n\}}{P\{N(t) = n\}} \\
&= \frac{(P\{1 \text{ event over } [0, \epsilon]\})^n \, P\{0 \text{ events over}(n\epsilon, t]\}}{P\{N(t) = n\}} \\
&= \frac{(\lambda\epsilon)^n e^{-n\lambda\epsilon}, (\lambda(t - n\epsilon))^0 \, e^{-\lambda(t-n\epsilon)}}{(\lambda t)^n e^{-\lambda t}/(n!)} \\
&= \frac{n!}{t^n}\epsilon^n
\end{aligned}
$$

Conditional Distribution of Arrival Times

- what we have got for $f(\boldsymbol{s} \mid N(t) = n)$?

$$f(\boldsymbol{s} \mid N(t) = n) = \frac{n!}{t^n} \tag{1}$$

- if the $Y_i,\ i = 1, ..., n$, are independent uniform RVs over $[0, t]$, then the joint density function of the order statistics $Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)}$ is given by [5]

$$g(y_1, y_2, \ldots, y_n) = \frac{n!}{t^n}, \qquad 0 < y_1 < y_2 < \cdots < y_n < t \tag{2}$$

- compare (1) and (2), i.e.,
  - conditioned on that $N(t) = n$, the time of the arrivals have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $[0, t]$

[5] See p. 317 of [Ros-3] or p. 276, § 6.6 [Ros-2].