# Traffic Engineering
# EN3370

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# Lecture 2: Exponential Distribution

# Exp. Distribution: Relevance to Traffic

- used when time interval modelling
- the time until the next arrival
- time until a customer completes service
- closely connected with the Poisson process
  - Poisson process is used to model customer arrivals
- fundamental to the theory of continuous-(time) Markov chains [1]

[1] the basis for queueing models (discussed later in the lectures)

# Exponential Distribution

- content of this discussion is mainly from [Ross-3, § 5]
- of course you are welcome to check the related material from other course references as well, e.g., [Ross-1, § 1.6]
- continuous random variable (RV) $X$ is said to have an exponential distribution with positive parameter $\lambda$, $\lambda > 0$, if the probability density function (pdf) $f(x)$ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

# CDF and CCDF of Exponential RV

- cumulative distribution function (CDF)

$$F_X(x) = \begin{cases} P\{X \leq x\} = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- complementary cumulative distribution function (CCDF)

$$\bar{F}_X(x) = \begin{cases} P\{X > x\} = e^{-\lambda x} & x \geq 0 \\ 1 & \text{otherwise} \end{cases}$$

# Mean and Variance of Exponential RV

- mean

$$\mathbb{E}\{X\} = \frac{1}{\lambda}$$

- variance

$$\mathtt{Var}\{X\} = \frac{1}{\lambda^2}$$

# Characteristic/Moment Generating Functions

- characteristic function $\phi_X$

$$\phi_X(u) = \mathbb{E}\{e^{juX}\} = \frac{\lambda}{\lambda - jt}$$

- moment generating function $M_X$

$$M_X(v) = \mathbb{E}\{e^{vX}\} = \frac{\lambda}{\lambda - v}, \quad v < \lambda$$

# Properties of Exponential Distribution

# Memoryless Property: Definition

- a random variable $X$ is said to be without memory or memoryless, if

$$P\{X > s + t \mid X > t\} = P\{X > s\} \tag{1}$$

  for all $s, t \geq 0$

- verify that (1) is equivalent to

$$\bar{F}(t + s) = \bar{F}(t)\bar{F}(s)$$

  for all $s, t \geq 0$

- show that exponential RVs are memoryless

# Memoryless Property: Interpretation

- suppose $X$ is the lifetime of some instrument `A`
- the memoryless property states:
  - probability that `A` survives for at least $s + t$ hours, given that it has already survived $t$ hours is the same as the probability that `A` survives for at least $s$ hours
  - if `A` is surviving at time $t$ then the distribution of its remaining lifetime is the original lifetime distribution

# Memoryless Property: Interpretation

- suppose $X$ is the time until some event occurs
- e.g., the time until the next bus arrives
  - Bob waited from $t = 0$ to $t = t$ time units
  - suppose, still no bus has arrived
  - Alice has just arrived at $t = t$
  - Bob's remaining waiting time distribution is identical to Alice's remaining waiting time distribution
  - thus, the remaining waiting time has no memory of previous waiting

# The Failure (or Hazard) Rate Function

- consider a continuous random variable $X$ having distribution function $F_X$ and density $f_X$
- the failure rate function $\lambda(t)$ is defined by

$$\lambda(t) = \frac{f_X(t)}{1 - F_X(t)}$$

- show that

$$\lambda(t) = \lambda$$

for an exponential RV with parameter $\lambda$

# Failure Rate: Interpretation

- suppose $X$ is the lifetime of some item
- given that the item has survived for $t$ hours
- we want to get an idea of the probability that it will not survive for an additional time $dt$
- i.e., $P\{X < t + dt \mid X > t\}$

# Failure Rate: Interpretation

- Thus we have

$$\begin{aligned} P\{X < t + dt \mid X > t\} &= P\{X \in (t, t + dt) \mid X > t\} \\ &= \frac{P\{X \in (t, t + dt), X > t\}}{P\{X > t\}} \\ &= \frac{P\{X \in (t, t + dt)\}}{P\{X > t\}} \\ &\approx \frac{f(t)dt}{1 - F(t)} = \lambda(t)dt \end{aligned}$$

- i.e., $\lambda(t)$ represents kind of a conditional probability intensity [2]

[2] compare with [Wozencraft, equation (2.38a)]

# Failure Rate to Compute the Distribution

- the failure rate function $\lambda(t)$ can be used to determines the distribution $F(t)$
- show that

$$F(t) = 1 - \exp\left\{-\int_0^t \lambda(\tau)d\tau\right\}$$

# Further Properties of Exponential Distribution

# Sum of IID Exponential RVs

- If $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables with parameter $\lambda$, the pdf $f_Z$ of $z = \sum_{i=1}^{n} X_i$ is given by

$$f_Z(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

- $f_Z$ is called the gamma distribution with parameters $n$ and $\lambda$, see § 5.6.1 of [Ross-2]

- proof ?

# $P\{X_1 < X_2\}$

- $X_1$ and $X_2$ are independent exponential random variables with parameters $\lambda_1$ and $\lambda_2$, respectively
- determine the probability that one RV is smaller than the other, i.e., $P\{X_1 < X_2\}$

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \tag{2}$$

- proof ?
- what can we say about $P\{X_1 = \min\{X_1, X_2\}\}$ ?

# Distribution/Density of the Minimum

- Let $X_1, \ldots, X_n$ be independent exponential random variables with rates $\lambda_1, \ldots, \lambda_n$, respectively. Then the random variable $X_{\min} = \min\{X_1, \ldots, X_n\}$ is exponentially distributed with rate $\lambda = \lambda_1 + \ldots + \lambda_n$
  - proof ?
  - what if $X_i$s are identical?

# Distribution/Density of the Maximum

- Let $X_1, \ldots, X_n$ be independent exponential random variables with rates $\lambda_1, \ldots, \lambda_n$, respectively. Let the random variable $X_{\max} = \max\{X_1, \ldots, X_n\}$.
- work out the CDF and pdf
- what if $X_i$s are identical?

# Some Properties of Exponential RV

- Let $X_1, \ldots, X_n$ be independent exponential random variables with rates $\lambda_1, \ldots, \lambda_n$, respectively. Then

$$P\{X_i = \min\{X_1, \ldots, X_n\}\} = \frac{\lambda_i}{\lambda_1 + \ldots + \lambda_n}$$

  - work out the result
    - use results in pages 15 and 16
  - what if $X_i$s are identical?

# Typical Units of Exponential RV $X$

- in this module RV $X$ typically represents a quantity of time
- parameter $\lambda$ represents a rate, in particular, events per time
- thus, $\mathbb{E}\{X\} = 1/\lambda$ is of units time