Summary

Markov chains

- DTMT
  - HOMOGENEOUS
  - Nonhomogeneous
- CTMC
  - Homogeneous
  - Nonhomogeneous

	HOMOGENEOUS	Nonhomogeneous	Homogeneous	Nonhomogeneous
$p_{ij}^{(n)}$, $P_{ij}(t)$	p_{ij}	$p_{ij}^{(n)}$	$p_{ij}(t)$	$p_{ij}(s,t)$
forward backward equations	$H[n-m] = H[n-1-m]\,P$ $P^{n-m} = P^{(n-1)-m}\,P$ $P^{n-m} = P\,P^{n-(m+1)}$	$H[m,n] = H[m,n-1]\,P[n-1]$ $H[m,n] = P[m]\,H[m+1,n]$	$\frac{d}{dt}H(t) = H(t)\,Q$ $H(t) = e^{Qt}$	$\frac{\partial}{\partial t}H(s,t) = H(s,t)\,Q(t)$ $H(s,t) = e^{\int_s^t Q(u)\,du}$
Transient behaviour	$\pi^{(n)} = \pi^{(n-1)}P$ $\pi^{(n)} = \pi^{(0)}P^n$	$\pi^{(n)} = \pi^{(n-1)}P[n-1]$ $\pi^{(n)} = \pi^{(0)}P[0]P[1]\cdots P[n-1]$	$\frac{d}{dt}\pi(t) = \pi(t)Q$ $\pi(t) = \pi(0)e^{Qt}$	$\frac{\partial}{\partial t}\pi(t) = \pi(t)Q(t)$ $\pi(t) = \pi(0)e^{\int_0^t Q(u)\,du}$
Limiting behaviour	$\pi = \pi P$, $1^T\pi = 1$	did not discuss	$\pi Q = 0$, $1^T\pi = 1$	did not discuss

For this to hold, DTMC should satisfy certain conditions

e.g., irreducible, recurrent, nonnull, aperiodic → Ergodic

For this to hold, DTMC should satisfy certain conditions

e.g., Embedded MC of the CTMC should satisfy some conditions, irreducibility, recurrency, nonnull aperiodicity

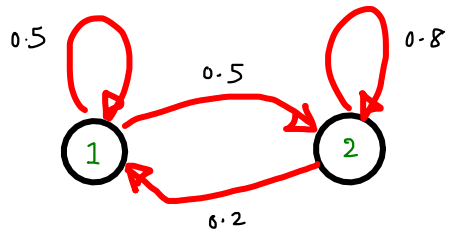
see [Ross-1] section 4.8, 5.5

★ Steps:

(1) Identify the states

(2) Identify the transitions

(3) Identify the transition probabilities



$p_{11}^{(2)} = (0.5)(0.5) + (0.2)(0.5) = 0.35$

$p_{12}^{(2)} = (0.5)^2 + (0.5)(0.8) = 0.65$

$p_{21}^{(2)} = (0.8)(0.2) + (0.2)(0.5) = 0.26$

$p_{22}^{(2)} = (0.8)^2 + (0.2)(0.5) = 0.74$

n	0	1	2	7	100
$p_{11}^{(n)}$	1	0.5	0.35	0.2859	0.2857
$p_{12}^{(n)}$	0	0.5	0.65	0.7141	0.7143
$p_{21}^{(n)}$	0	0.2	0.26	0.2857	0.2857
$p_{22}^{(n)}$	1	0.8	0.74	0.7143	0.7143

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}, \quad P^{(2)} = P^2 = \begin{pmatrix} 0.35 & 0.65 \\ 0.26 & 0.74 \end{pmatrix}$$

$$P^{(7)} = P^7 = \begin{pmatrix} 0.2859 & 0.7141 \\ 0.2857 & 0.7143 \end{pmatrix}, \quad P^{(100)} = P^{100} = \begin{pmatrix} 0.2857 & 0.7143 \\ 0.2857 & 0.7143 \end{pmatrix}$$

Note: the above DTMC is irreducible and Ergodic $\rightarrow$ $\pi P = \pi$, $\pi \mathbf{1} = 1$

$0.5\pi_1 + 0.2\pi_2 = \pi_1$ $\qquad$ $\pi_1 + \pi_2 = 1$

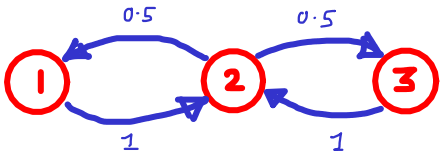
$0.5\pi_1 + 0.2(1-\pi_1) = \pi_1$

$0.7\pi_1 = 0.2 \rightarrow \pi_1 = 2/7$ thus $\pi_2 = 5/7$

The above chain is nice in the sense { ① Probabilities converged ② Initial state is forgotten

When will this happen? Lets see 2 Examples now

Example 1



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$p_{11}^{(0)} = 1, \quad p_{12}^{(0)} = 0, \quad p_{13}^{(0)} = 0$

$p_{21}^{(0)} = 0, \quad p_{22}^{(0)} = 1, \quad p_{23}^{(0)} = 0$

$p_{31}^{(0)} = 0, \quad p_{32}^{(0)} = 0, \quad p_{33}^{(0)} = 1$

$$P^{(2)} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

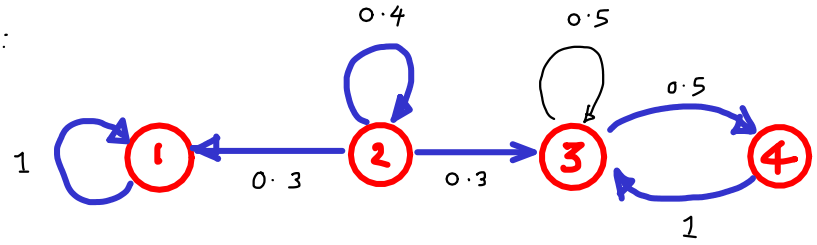
$$P^{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{(n)} = P^n = \begin{cases} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, & n \text{ even} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}, & n \text{ odd} \end{cases}$$

$\lim_{n\to\infty} P^n$ DOES NOT CONVERGE ! the first property in above doesn't hold

PERIODICITY is the issue here

* Example 2 :



$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} P^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 1/3 & 1/6 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

Discuss these probabilities

$\lim_{n\to\infty} P^n$ CONVERGES. But converging probabilities depends on the initial state ! the 2nd property above doesn't hold !

REDUCIBILIY is the issue here

* Check page 27 of Lecture slides set 4.

Irreducible AND aperiodic $\Longrightarrow$ $\pi_j = \lim_{n\to\infty} \pi_j^{(n)}$ exists

independent of initial distribution

(recall $\pi^{(n)} = \pi^{(0)} P^n$)

↳ i.e.,

$\lim_{n\to\infty} P^n$ exists

* Recurrent states: $f_{jj} = 1 = P[\text{ever returning to } j \text{ after leaving from } j]$

↳ starting from j, whenever you go, $\exists$ a way of returning to j

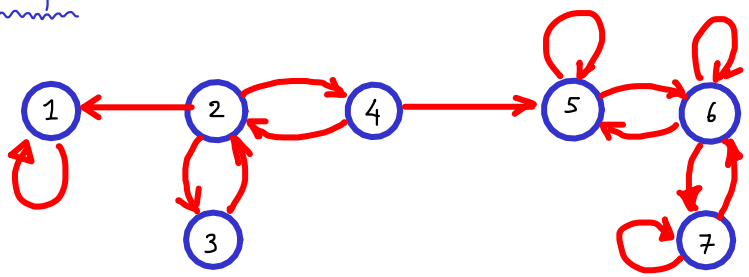
Transient state: If not recurrent, i.e., $f_{jj} < 1$

↳ j is visited only finite # of times.

Note: $P[X_n = j] \longrightarrow 0$

Irreducible + aperiodic + finite # of stets $\Rightarrow$ The MC is (+)ve recurrent

Example



3 communicating classes

{1} {2,3,4} {5,6,7}

1 → recurrent (R)
2 → transient (T)
3 → T
4 → T
5 → R
6 → R
7 → R

recurrence is a class property

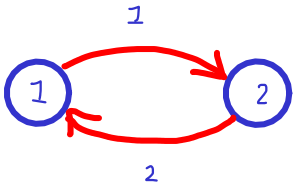
transience is a class property

recurrence is a class property

*** INITIAL CONDITIONS MATTER IN THE LONG RUN
If you have more than one recurrent state.

* Periodic states

- states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that the transition from one group lead to the next group.



$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

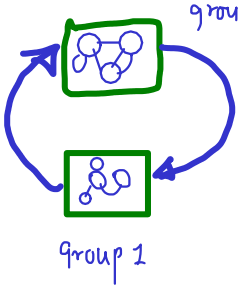
$$P^{(2)} = P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

⋮

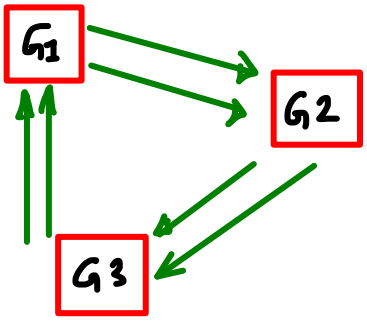
e.g.,

transitions from Group 1 → Group 2

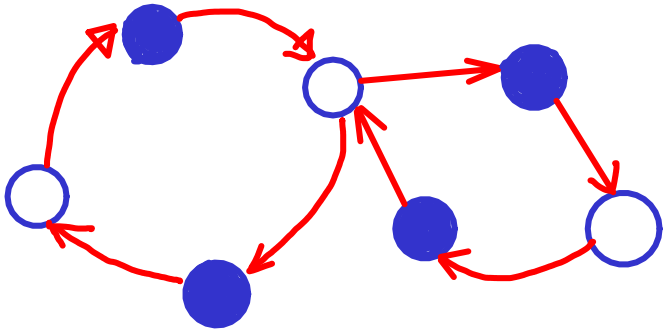


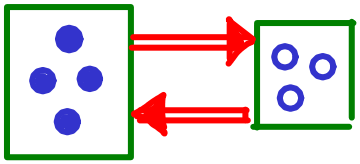
transitions from Group 2 → Grop 1

period is 2



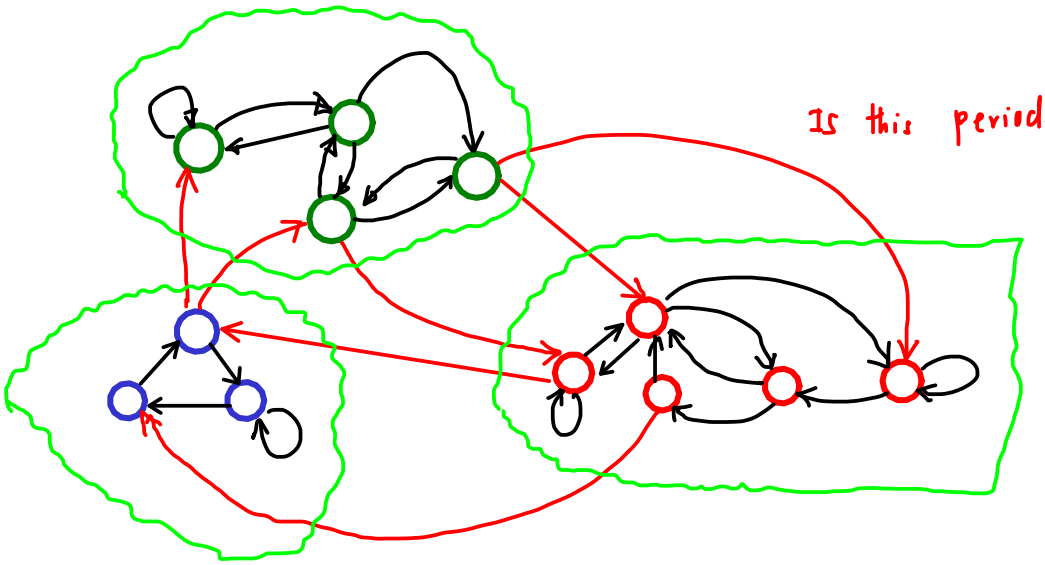
period is 3





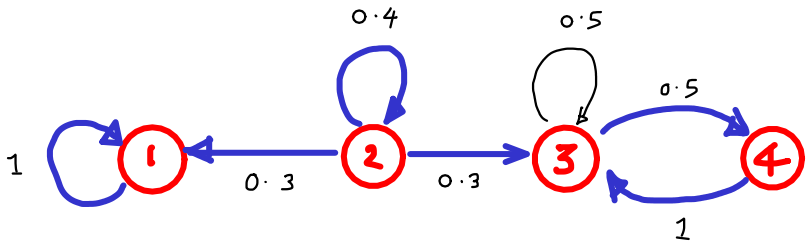
period is 2

** Just by adding a self-loop to at least one of the node, we can make the M.C aperiodic



Is this periodic ? NO

* Computing Conditional probabilities



$$P[X_3 = 4, X_2 = 3, X_1 = 2 \mid X_0 = 2]$$

$$= P[X_3 = 4, X_2 = 3 \mid X_1 = 2, X_0 = 2]\, P[X_1 = 2 \mid X_0 = 2]$$

$$= P[X_3 = 4 \mid X_2 = 3, X_1 = 2, X_0 = 2]\, P[X_2 = 3 \mid X_1 = 2, X_0 = 2]\, P[X_1 = 2 \mid X_0 = 2]$$

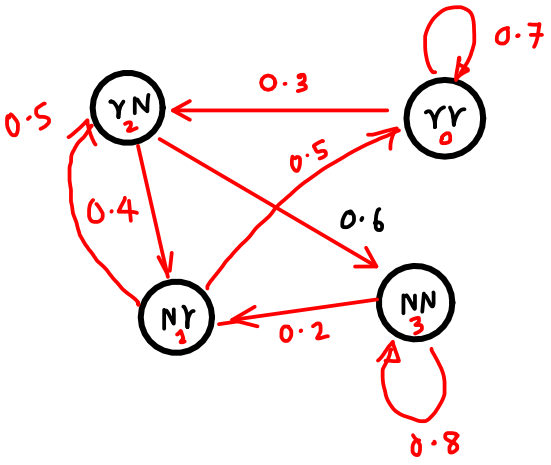
$$= P[X_3 = 4 \mid X_2 = 3]\, P[X_2 = 3 \mid X_1 = 2]\, P[X_1 = 2 \mid X_0 = 2]$$

$$= p_{34}\, p_{23}\, p_{22} = p_{22}\, p_{23}\, p_{34} \;//$$

$$P[X_3 = 4 \mid X_0 = 2] = p_{24}^{(3)} = p_{23}\, p_{33}\, p_{34} + p_{22}\, p_{23}\, p_{34} \;//$$

★ Ross-2014, Example 4.4, 185

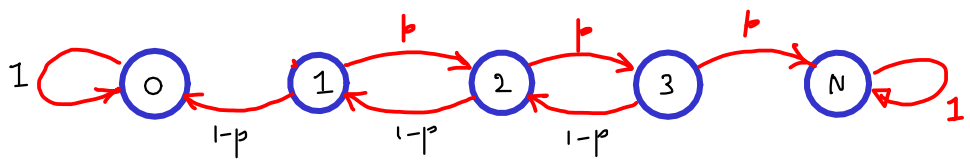
	Yest.	today	tomorrow	
0	Y	Y	Y (0.7)	NY (0.3)
2	Y	NY	Y (0.4)	NY (0.6)
1	NY	Y	Y (0.5)	NY (0.5)
3	NY	NY	Y (0.2)	NY (0.8)



$$P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

★ Gambler's Ruin [Ross-16, Example 4.6; Durrett-16, Example 1.1]

You play gambling. { In any turn you play you win with 1 $ with probability 0.4. lose $1 w.p 0.6. Rule: quit playing when fortune reaches $N. If fortune reaches $0, Casino sucks, you stop.



$p = 0.4$

$1-p = 0.6$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \lim_{n \to \infty} P^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.876 & 0 & 0 & 0 & 0.1231 \\ 0.693 & 0 & 0 & 0 & 0.3077 \\ 0.4154 & 0 & 0 & 0 & 0.5846 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

★ Social mobility [Durrett-16, Example 1.9]

X_n : families social comfort in n^{th} generation, either low, middle, upper

ie 1 2 3

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}, \quad \lim_{n \to \infty} P^n = \begin{bmatrix} 0.4681 & 0.3404 & 0.1915 \\ 0.4681 & 0.3404 & 0.1915 \\ 0.4681 & 0.3404 & 0.1915 \end{bmatrix}$$

{ you will get the same if $\pi P = \pi$, $\pi^T 1 = 1$ is solved

Branch Preference [Durrett-16, Example 1.5]

X_n : Product preference in n^{th} choice # of products say 3

$i = 1, 2, 3$

$$P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

①

customer arrivals into a shop

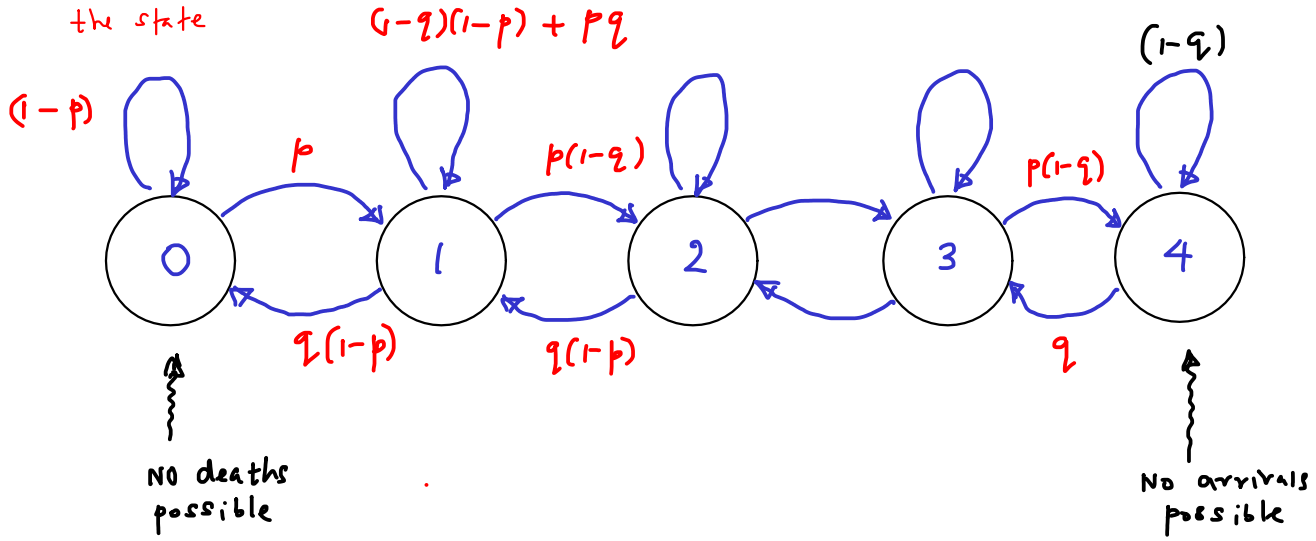
arrivals ~ Bernoulli (p)

↳ i.e., geometric inter-arrival times

Customer Service ~ Bernoulli (p) times

X_n = # of customers at epoch n

↑ the state



②

customer arrivals into a shop

arrivals ~ Poisson with parameter λ

↳ i.e., exponential inter-arrival times

Customer Service ~ Exponential times

$X(t)$ = # of customers at time t

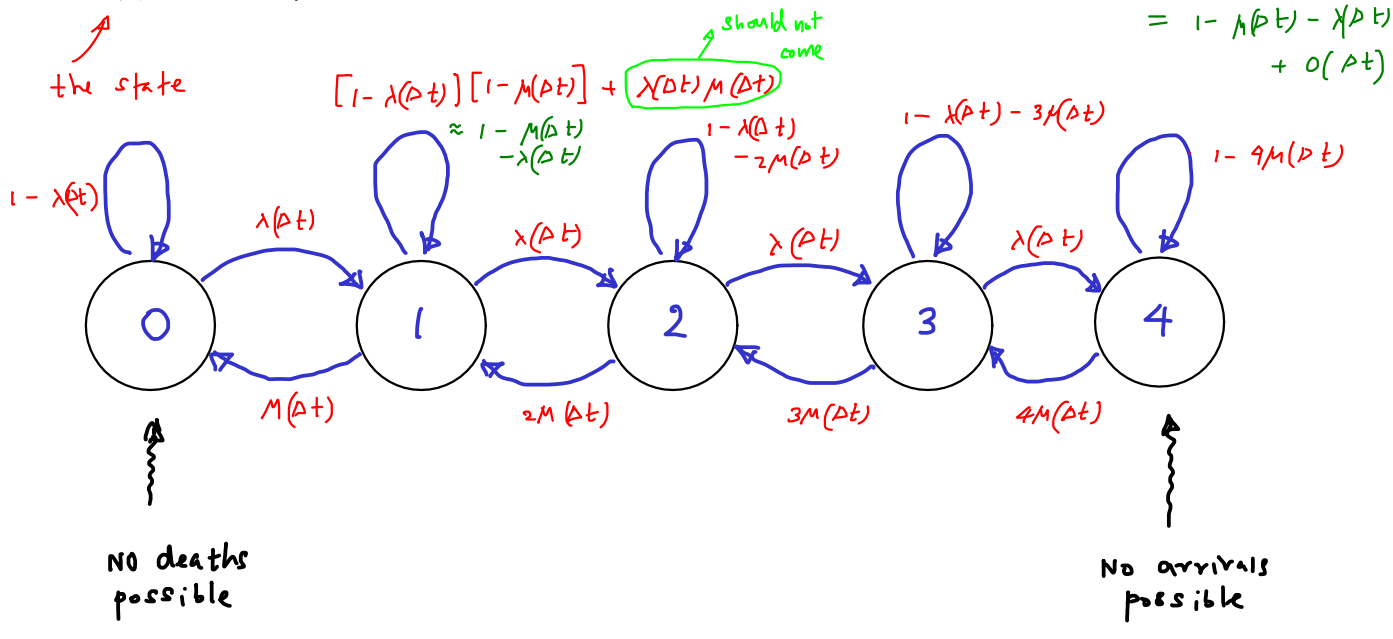
↑ the state

$[1-\lambda(\Delta t)][1-\mu(\Delta t)] = 1 - \mu(\Delta t) - \lambda(\Delta t) + \lambda\mu(\Delta t)^2$

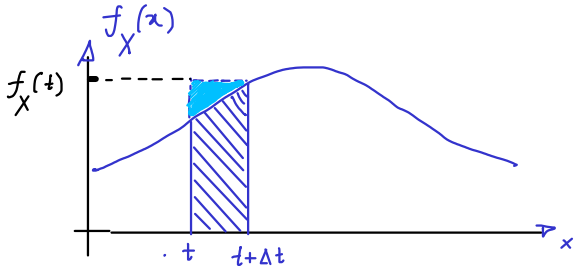
$= 1 - \mu(\Delta t) - \lambda(\Delta t) + o(\Delta t)$

$[1-\lambda(\Delta t)][1-\mu(\Delta t)] + \lambda(\Delta t)\mu(\Delta t)$ ← should not come

$\approx 1 - \mu(\Delta t) - \lambda(\Delta t)$







Consider X a RV which denote the time between adjacent arrivals.

CDF = $F_X(x) = P[X \le x]$

pdf = $f_X(x)$

$f_X(t)(\Delta t) = P[X \in [t, t+\Delta t]]$

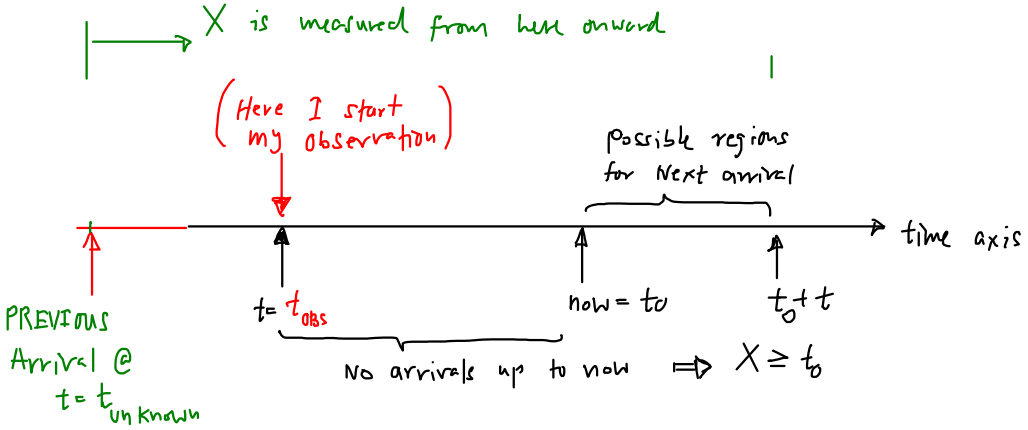
$= P[$next arrival occurs at least t and at most $(t+\Delta t)$ from the time of the last arrival$]$

suppose $f_X(x) = \lambda e^{-\lambda x}$, i.e., Exponential distribution

thus $F_X(x) = 1 - e^{-\lambda x}$

$P[$next arrival occurs t seconds from NOW $\mid$ no arrivals up to now$]$ —— (1)

($t = t_0$ for NOW; $t = t_0$ for up to now)



(1) is equivalent to $P[X \le t+t_0 \mid X \ge t_0] = \dfrac{P[X \le t+t_0,\ X \ge t_0]}{P[X \ge t_0]}$

$= \dfrac{P[X \in [t_0, (t+t_0)]]}{e^{-\lambda t_0}} = \dfrac{F_X(t+t_0) - F_X(t_0)}{e^{-\lambda t_0}}$

$= \dfrac{[1 - e^{-\lambda(t+t_0)}] - [1 - e^{-\lambda t_0}]}{e^{-\lambda t_0}} = (1 - e^{-\lambda t})$

$= F_X(t)$

- Thus, the distribution of the remaining time until the next arrival is exponentially distributed with parameter λ

Position ourself at t_0 (i.e., what ever the time now). What is the probability that $\exists$ an arrival within next Δt ?

given that no arrival up to t_0

$$= \quad 1 - e^{-\lambda(\Delta t)} \quad = \quad 1 - \left[1 - \lambda(\Delta t) + \frac{\lambda^2(\Delta t)^2}{\lfloor 2} - \frac{\lambda^3(\Delta t)^3}{\lfloor 3} + \cdots \right]$$

$$= \quad \lambda(\Delta t) \; + \; o(\Delta t)$$

Probability that ∄ no arrivals within next Δt, given that no arrivals up to t_0

$$P[X > t_0 + \Delta t \mid X > t_0] = 1 - \lambda(\Delta t) + o(\Delta t)$$

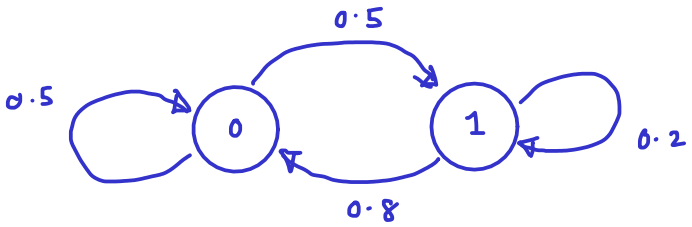
Probability that ∃ two or more arrivals in next (Δt), given that no arrivals up to t_0

$$= 1 - \underbrace{P[\nexists \text{ no arrivals in } \Delta t]}_{} - P[\exists \text{ 1 arrival in } \Delta t]$$

$$= 1 - [1 - \lambda(\Delta t) + o(\Delta t)] - [\lambda(\Delta t)]$$

$$= 1 - 1 + \lambda(\Delta t) + o(\Delta t) - \lambda(\Delta t) = o(\Delta t)$$

3



$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{bmatrix}$$

apply transient equation

$$\pi^{(n)} = \pi^{(n-1)} P \quad \text{——} \; ①$$

Define : $\Pi(z) \triangleq \sum_{n=0}^{\infty} \pi^{(n)} z^n$ —— ✱ $\quad \Pi(z) \longleftrightarrow \pi^{(n)}$

Multiply ① by z^n $\longrightarrow$ $\underbrace{\sum_{n=1}^{\infty} \pi^{(n)} z^n}_{} = \sum_{n=1}^{\infty} \pi^{(n-1)} z^n P = z \sum_{n=1}^{\infty} \pi^{(n-1)} z^{n-1} P$

$$\Pi(z) - \pi^{(0)} = z \Pi(z) P$$

$$\pi(z) [I - zP] = \pi^{(0)}$$

$[I - zP]^{-1} \longleftrightarrow P^n$

$$\Pi(z) = \pi^{(0)} [I - zP]^{-1} \quad \text{——} \; ✱$$

$$\Pi(z) = \pi^{(0)} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z \begin{pmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{pmatrix} \right]^{-1}$$

$$= \pi^{(0)} \begin{bmatrix} (1 - 0.5z) & -0.5z \\ -0.8z & 1 - 0.2z \end{bmatrix}^{-1}$$

$$= \frac{\pi^{(0)}}{(\det \text{ Matrix})} \begin{bmatrix} 1-0.2z & 0.5z \\ 0.8z & 1-0.5z \end{bmatrix}$$

$$= \frac{\pi^{(0)}}{[1-0.5z][1-0.2z] - 0.4z^2} \begin{bmatrix} 1-0.2z & 0.5z \\ 0.8z & 1-0.5z \end{bmatrix}$$

$$\Pi(z) = \pi^{(0)} \begin{bmatrix} \frac{1-0.2z}{1-0.7z-0.3z^2} & \frac{0.5z}{1-0.7z-0.3z^2} \\ \frac{0.8z}{1-0.7z-0.3z^2} & \frac{1-0.5z}{1-0.7z-0.3z^2} \end{bmatrix}$$

$z_1 = 1$, $z_2 = -\frac{10}{3}$

$$= \pi^{(0)} \begin{bmatrix} \left(\frac{-0.6154}{(z-z_1)} + \frac{1.2821}{(z-z_2)}\right) & \left(\frac{-0.3846}{(z-z_1)} + \frac{-1.2821}{(z-z_2)}\right) \\ \left(\frac{-0.6154}{(z-z_1)} + \frac{-2.0513}{(z-z_2)}\right) & \left(\frac{-0.3846}{(z-z_1)} + \frac{2.0513}{(z-z_2)}\right) \end{bmatrix}$$

$$\Pi(z) = \frac{\pi^{(0)}}{z_1(1-(z_1^{-1})z)} \underbrace{\begin{bmatrix} 0.6154 & 0.3846 \\ 0.6154 & 0.3846 \end{bmatrix}}_{A} + \frac{\pi^{(0)}}{z_2[1-(z_2^{-1})z]} \underbrace{\begin{bmatrix} -1.2821 & 1.2821 \\ 2.0513 & -2.0513 \end{bmatrix}}_{B}$$

z^{-1} ↓

$$\pi^{(n)} = \frac{\pi^{(0)}}{z_1} A + \frac{\pi^{(0)}}{z_2} (z_2^{-1})^n B \qquad (z_2^{-1}) = \left(-\frac{3}{10}\right)$$

$$\pi^{(n)} = \pi^{(0)} A - \frac{3}{10}\pi^{(0)} (-0.3)^n B$$

when $n \to \infty$ $\quad \pi^{(n)} = \pi^{(0)} (A \to P^{\infty})$ } verify !!

(4) For the same MC above in (3) Compute the limiting behaviour

$$\pi = \pi \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{bmatrix} \Rightarrow \quad \pi_1 = 0.5\pi_1 + 0.8\pi_2 \text{ — (1)}$$

$$\pi_2 = 0.5\pi_1 + 0.2\pi_2$$

$$1^T\pi = 1 \qquad \pi_2 = 1 - \pi_1 \text{ — (2)}$$

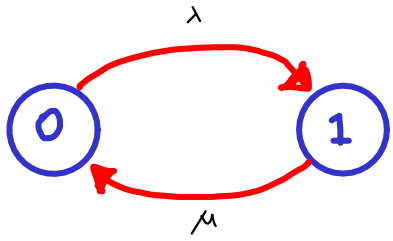
① and ② above $\Rightarrow$ $\pi_1 = 0.5\pi_1 + (0.8)(1-\pi_1)$

$\pi_1(1-0.5+0.8) = 0.8$

$\pi_1 = \frac{0.8}{1.3} = 0.6154$

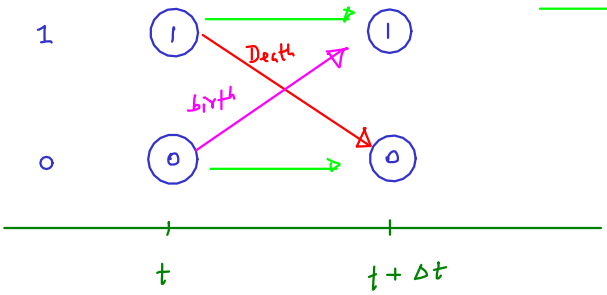
Thus, $\pi_2 = 1 - 0.6154 = 0.3846$ //

5



$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

* Here further arrivals are not allowed ⇝ 1
* Here deaths are not allowed ⇝ 0



Note that the arrows drawn in green colour should carefully be handled when writing the equations below. The reason is that for instance we have only one server in the system and no queues. See * in the front of the figure here.

recall: Homogeneous CTMC:

$$\pi_0(t+\Delta t) = \pi_1(t)\, p_{10}(\Delta t) + \pi_0(t)\, p_{00}(\Delta t)$$

** This is not present

$$= \pi_1(t)\left[(\Delta t)\mu + o(\Delta t)\right] + \pi_0(t)\left[1-(\Delta t)\mu + o(\Delta t)\right] \times \left[1-(\Delta t)\lambda + o(\Delta t)\right]$$

$$= \mu\pi_1(t)(\Delta t) + \pi_0(t)\left[1-(\Delta t)\lambda\right] + o(\Delta t)$$

$$\pi_0(t+\Delta t) = \mu\pi_1(t)(\Delta t) + \pi_0(t)\left[1-(\Delta t)\lambda + o(\Delta t)\right] + o(\Delta t)$$

$$= \pi_0(t) - \lambda\pi_0(t)(\Delta t) + \mu\pi_1(t)(\Delta t) + o(\Delta t)$$

divide by (Δt):

$$\frac{\left[\pi_0(t+\Delta t) - \pi_0(t)\right]}{(\Delta t)} = \frac{\left[\mu\pi_1(t) - \lambda\pi_0(t)\right](\Delta t)}{(\Delta t)} + \frac{o(\Delta t)}{(\Delta t)}$$

$\lim_{(\Delta t)\to 0}$

$$\frac{d}{dt}\pi_0(t) = -\lambda\pi_0(t) + \mu\pi_1(t)$$

similarly $\pi_1(t+\Delta t) = \pi_1(t)\, p_{11}(\Delta t) + \pi_0(t)\, p_{01}(\Delta t)$

This is not there!! Be careful!!

$$= \pi_1(t)\left[1-\mu(\Delta t)+o(\Delta t)\right]\left[1-\lambda(\Delta t)+o(\Delta t)\right] + \pi_0(t)\left[\lambda(\Delta t)+o(\Delta t)\right]$$

$$\pi_1(t+\Delta t) = \pi_1(t)\left[1 - \mu(\Delta t) + o(\Delta t)\right] + \lambda\,\pi_0(t)(\Delta t) + o(\Delta t)$$

$$\frac{\left[\pi_1(t+\Delta t) - \pi_1(t)\right]}{(\Delta t)} = \frac{-\mu\,\pi_1(t)(\Delta t)}{(\Delta t)} + \frac{\lambda\,\pi_0(t)(\Delta t)}{(\Delta t)} + \frac{o(\Delta t)}{(\Delta t)}$$

$$\lim_{(\Delta t)\to 0} \quad \frac{d}{dt}\pi_1(t) = -\mu\,\pi_1(t) + \lambda\,\pi_0(t)$$

$$\frac{d}{dt}\pi_0(t) = -\lambda\,\pi_0(t) + \mu\,\pi_1(t)$$

$$\frac{d}{dt}\pi_1(t) = -\mu\,\pi_1(t) + \lambda\,\pi_0(t)$$

($-\lambda$: q_{00}; μ: q_{10}; $-\mu$: q_{11}; λ: q_{01})

So we have to solve these two equations

we could have directly apply the "Transient Behaviour" equations we have derived [A big hammer]

↳ see page (28) of lecture 5 slides ▽

to yield the same result

$$\text{i.e.,} \quad \frac{d}{dt}\pi(t) = \pi(t)\,Q \Rightarrow \frac{d}{dt}\left[\pi_0(t) \;\; \pi_1(t)\right] = \left[\pi_0(t) \;\; \pi_1(t)\right]\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

↓ solution is

$$\pi(t) = \pi(0)\,e^{Qt}$$

$$\left[\pi_0(t) \;\; \pi_1(t)\right] = \left[\pi_0(0) \;\; \pi_1(0)\right] e^{Qt}$$

By matlab: $\lim_{t\to\infty} \left[\pi_0(t) \;\; \pi_1(t)\right] = \left[\pi_0(0) \;\; \pi_1(0)\right]\left[\lim_{t\to\infty} e^{Qt}\right]$

type expm(Q*75) in matlab ← i.e., t=75 You are already there ▽

$$= \left[\pi_0(0) \;\; \pi_1(0)\right]\begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$= \left[\frac{2}{3} \;\; \frac{1}{3}\right] //$$

assumed $\lambda = 0.1$, $\mu = 0.2$

Note: $Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$

Alternatively $\det\left[sI - Q\right] \Rightarrow \det\begin{bmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{bmatrix} = (s+\lambda)(s+\mu) - \lambda\mu$

Characteristic polynomial $\Rightarrow$ $s^2 + (\lambda+\mu)s = 0$

$s(s+(\lambda+\mu)) = 0 \Rightarrow$ eigen values are $s = 0, -(\lambda+\mu)$ //

$$\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{aligned} -\lambda v_1 + \lambda v_2 - \lambda v_1 &= 0 \\ \mu v_1 - \mu v_2 - \lambda v_2 &= 0 \end{aligned}$$

$$\left[\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} + \begin{pmatrix} (\lambda+\mu) & 0 \\ 0 & (\lambda+\mu) \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} \mu & \lambda \\ \mu & \lambda \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -\mu \end{pmatrix} \qquad V = \begin{pmatrix} 1 & \lambda \\ 1 & -\mu \end{pmatrix}$$

$$0, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad -(\lambda+\mu), \frac{1}{\sqrt{\lambda^2+\mu^2}}\begin{pmatrix} -\lambda \\ +\mu \end{pmatrix}$$

(eig value, eig vector) pairs

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-\lambda}{\sqrt{\lambda^2+\mu^2}}, \frac{\mu}{\sqrt{\lambda^2+\mu^2}}\right)$$

$$e^{Qt} = e^{V\Lambda V^{-1}t} = V e^{\Lambda t} V^{-1} = V\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} V^{-1}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -\lambda/\sqrt{\lambda^2+\mu^2} \\ 1/\sqrt{2} & \mu/\sqrt{\lambda^2+\mu^2} \end{bmatrix}$$

$$V^{-1} = \frac{1}{\det V}\begin{bmatrix} \mu/\sqrt{\lambda^2+\mu^2} & \lambda/\sqrt{\lambda^2+\mu^2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

eig value: 0 (↑) $\quad -(\lambda+\mu)$ (↑)

$$\det V = \frac{1}{\sqrt{2}}\frac{\mu}{\sqrt{\lambda^2+\mu^2}} + \frac{\lambda}{\sqrt{2}\sqrt{\lambda^2+\mu^2}}$$

$$\det V = \frac{1}{\sqrt{2}}\frac{(\lambda+\mu)}{\sqrt{\lambda^2+\mu^2}}$$

$$\Rightarrow V^{-1} = \begin{bmatrix} \sqrt{2}\frac{\mu}{(\lambda+\mu)} & \sqrt{2}\frac{\lambda}{(\lambda+\mu)} \\ -\frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} & \frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} \end{bmatrix}$$

$$\therefore e^{Qt} = \begin{bmatrix} 1/\sqrt{2} & -\lambda/\sqrt{\lambda^2+\mu^2} \\ 1/\sqrt{2} & \mu/\sqrt{\lambda^2+\mu^2} \end{bmatrix}\begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{bmatrix}\begin{bmatrix} \sqrt{2}\frac{\mu}{(\lambda+\mu)} & \sqrt{2}\frac{\lambda}{(\lambda+\mu)} \\ -\frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} & \frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} \end{bmatrix}$$

$$\lim_{t\to\infty} e^{Qt} = \begin{bmatrix} 1/\sqrt{2} & \frac{-\lambda}{\sqrt{\lambda^2+\mu^2}} \\ 1/\sqrt{2} & \frac{\mu}{\sqrt{\lambda^2+\mu^2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}\frac{\mu}{(\lambda+\mu)} & \sqrt{2}\frac{\lambda}{(\lambda+\mu)} \\ -\frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} & \frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}\frac{\mu}{(\lambda+\mu)} & \sqrt{2}\frac{\lambda}{(\lambda+\mu)} \\ -\frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} & \frac{\sqrt{\lambda^2+\mu^2}}{(\lambda+\mu)} \end{bmatrix} = \begin{bmatrix} \frac{\mu}{(\lambda+\mu)} & \frac{\lambda}{(\lambda+\mu)} \\ \frac{\mu}{(\lambda+\mu)} & \frac{\lambda}{(\lambda+\mu)} \end{bmatrix}$$

$$\therefore \quad \pi(t) = \begin{bmatrix} \pi_0(0) & \pi_1(0) \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \pi_0(t) & \pi_1(t) \end{bmatrix} = \begin{bmatrix} \frac{\mu}{(\lambda+\mu)} & \frac{\lambda}{(\lambda+\mu)} \end{bmatrix}$$

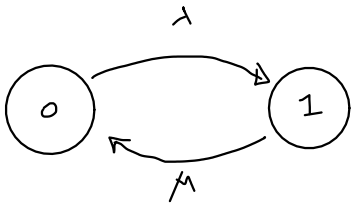
When $\lambda = 0.1$ and $\mu = 0.2$ we have,

$$\pi_0(t) = \frac{0.2}{(0.3)} = \frac{2}{3}$$

$$\pi_1(t) = \frac{0.1}{(0.3)} = \frac{1}{3}$$

} same as above! //

6 For the same MC above compute the limiting behaviour?



$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Solve $\pi Q = 0 \qquad 1^T\pi = 1$

$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} -\lambda\pi_0 + \mu\pi_1 &= 0 \\ \lambda\pi_0 - \mu\pi_1 &= 0 \end{aligned}\right\} \text{ Identical equations} \rightarrow \pi_0 = \frac{\mu\pi_1}{\lambda}$$

$$\therefore \frac{\mu\pi_1}{\lambda} + \pi_1 = 1$$

$\Rightarrow \quad (\lambda+\mu)\,\pi_1 = 1\cdot\lambda \quad \Rightarrow \quad \pi_1 = \dfrac{\lambda}{(\lambda+\mu)}$

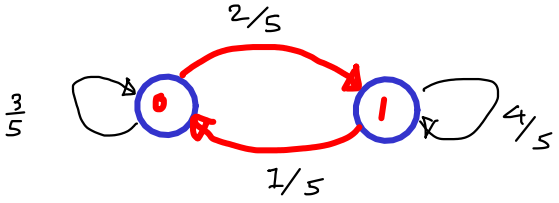
$\Rightarrow \quad \pi_0 = \dfrac{\mu}{(\lambda+\mu)}$

When $\lambda = 0.1$ and $\mu = 0.2$

$\hookrightarrow \quad \pi_0 = \dfrac{2}{3}, \quad \pi_1 = \dfrac{1}{3}$ //

Compare with the transient probabilities

7



$$P = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{bmatrix}$$

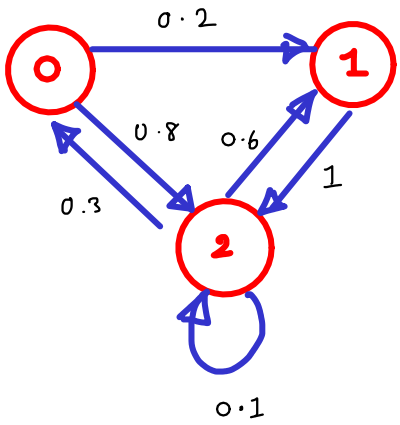
$$\lim_{n\to\infty} P^n = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix}$$

Note: $\pi_1 + \pi_2 = 1$

Note: the above DTMC is irreducible, aperiodic, recurrent nonnull (i.e., +ve recurrent)

Thus, the solⁿ for $\pi = [\pi_1 \ \pi_2]$ could be obtained by equations $\pi = \pi P$, $\pi 1 = 1$

8



$$P = \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0 & 0 & 1 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}$$

the DTMC is irreducible, aperiodic, +ve recurrent $\longrightarrow$

$\lim_{n\to\infty} P^n$ converges, given by $\pi P = \pi$, $\pi 1 = 1$

Solve $\pi P = \pi$ and $\pi 1 = 1$

$$\pi = [\pi_0 \quad \pi_1 \quad \pi_2] = [0.153 \quad 0.337 \quad 0.510]$$

Check

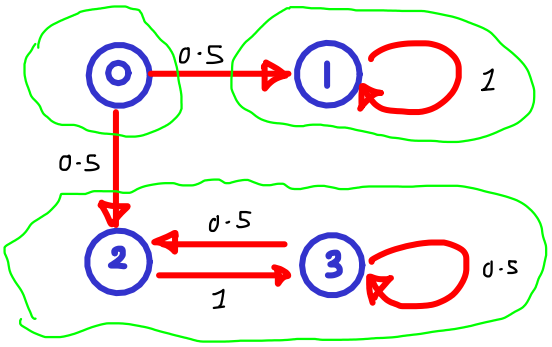
$$P^{10} = \begin{bmatrix} 0.1713 & 0.3689 & 0.4598 \\ 0.1858 & 0.3943 & 0.4199 \\ 0.1260 & 0.2891 & 0.5849 \end{bmatrix}$$

$$P^{50} = \begin{bmatrix} 0.1531 & 0.3368 & 0.5102 \\ 0.1531 & 0.3368 & 0.5102 \\ 0.1530 & 0.3367 & 0.5102 \end{bmatrix}$$

$$P^{40} = \begin{bmatrix} 0.1558 & 0.3416 & 0.5026 \\ 0.1580 & 0.3454 & 0.4966 \\ 0.1490 & 0.3258 & 0.5215 \end{bmatrix}$$

$$P^{100} = \begin{bmatrix} 0.1531 & 0.3367 & 0.5102 \\ 0.1531 & 0.3367 & 0.5102 \\ 0.1531 & 0.3367 & 0.5102 \end{bmatrix}$$

9



$$P = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Note: the chain is REDUCIBLE → there are 3 communicating classes. In particular, $\{0\}$, $\{1\}$, and $\{2,3\}$

Check $\pi_1 = \pi P$ $\quad \pi 1 = 1$

Just a try !

$\pi_0 = 0$

$\pi_1 = 0.5\,\pi_0 + \pi_1$

$\pi_2 = 0.5\,\pi_0 + 0.5\,\pi_3$

$\pi_3 = \pi_2 + 0.5\,\pi_3$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

Many solutions:

e.g., $\pi = [0 \;\; 0.7 \;\; 0.1 \;\; 0.2]$

$\pi = [0 \;\; 0.1 \;\; 0.3 \;\; 0.6]$

⋮

but

$$\lim_{n \to \infty} P^n = \begin{bmatrix} 0 & 1/2 & 1/6 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}$$

take class $\{2,3\}$

$$\bar{P} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} \rightarrow \bar{\pi}_1 = \bar{\pi}\,\bar{P} \qquad \bar{\pi}_2 + \bar{\pi}_3 = 1$$

$$[\bar{\pi}_2 \;\; \bar{\pi}_3] = [\bar{\pi}_2 \;\; \bar{\pi}_3] \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}$$

$\bar{\pi}_2 = 0.5\,\bar{\pi}_3$

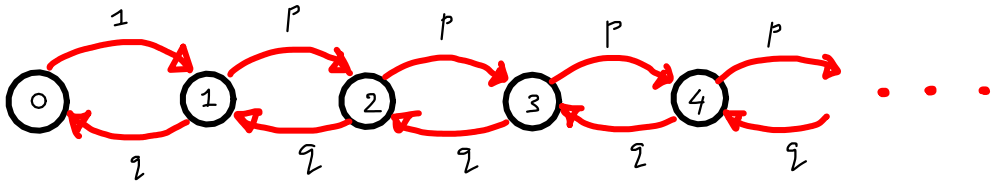
$\bar{\pi}_3 = \bar{\pi}_2 + 0.5\,\bar{\pi}_3$

$\bar{\pi}_2 = (0.5)(1 - \bar{\pi}_2) = 0.5 - 0.5\,\bar{\pi}_2$

$1.5\,\bar{\pi}_2 = 0.5$

$\bar{\pi}_2 = 1/3 \qquad \bar{\pi}_3 = 2/3$

(10)



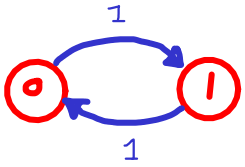
The DTMC above is <u>irreducible</u>, periodic [with period = 2]

But if $p > q$, then each state is transient. Intuition suggests that the chain is more likely to move to right

→ eventually the state drifts to ∞

that is $f_j < \infty \quad \forall j$

(11)



The Chain is irreducible. Periodic

Let us solve $\pi = \pi P$, $\pi 1 = 1$

$$[\pi_0 \quad \pi_1] = [\pi_0 \quad \pi_1]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\pi = [0.5 \quad 0.5]$ is a solution as our intuition suggests.

$$[0.5 \quad 0.5]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [0.5 \quad 0.5]$$

Let's check $\lim_{n \to \infty} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ = Do Not Converge

$$P^m = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & m \text{ is even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & m \text{ is odd} \end{cases}$$

Eventhough $\lim_{n \to \infty} P^n$ does not exist, $\lim_{n \to \infty} \pi^{(n)}$ exists if $\pi^{(0)}$ is appropriately Chosen.