Introduction to Telecommunications
EN1052

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Lecture 4: Spectral Content of a Signal

Fourier Transform

- recall: FT associates with each frequency f a *measure* of its contribution to the signal
- useful when the signal is transformed by a LTI system
  - the frequency components of the signal is independently weighted by the system transfer function
- this concept can be extended for signals, either discrete or continuous, deterministic or random [1]
  - energy associated with a frequency f
  - power associated with a frequency f

[1] Random signals are not discussed in detail in this introductory module.

Energy Density Spectral (EDS) of Deterministic Signals

Energy

- let $x(t)$ be a continuous deterministic signal
- the energy $\mathcal{E}_x$ of x is defined as

$$\mathcal{E}_x \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1}$$

  provided that the integral above is finite

- in the transformed domain, we have

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{2}$$

Energy Density Spectral

- the energy density spectral (EDS) $\mathcal{S}_x$ associated with x is defined as

$$\mathcal{S}_x(f) \triangleq |X(f)|^2 \tag{3}$$

Power Density Spectral (PDS) of Deterministic Signals

Power

- let $x(t)$ be a continuous deterministic signal whose energy is not finite
- the average power $\mathcal{P}_x$ of x is defined as

$$\mathcal{P}_x \triangleq \lim_{a\to\infty} \frac{1}{a} \int_{-a/2}^{a/2} |x(t)|^2 dt \tag{4}$$

provided that the integral above is finite

Power

- define the truncated signal

$$x_a(t) \triangleq \begin{cases} x(t), & -a/2 \leq t \leq a/2 \\ 0, & \text{otherwise} \end{cases}$$

- then $\mathcal{P}_x$ can be written as

$$\mathcal{P}_x = \lim_{a \to \infty} \frac{1}{a} \mathcal{E}_{x_a} \tag{5}$$

where $\mathcal{E}_{x_a}$ is the energy of x_a

Power Density Spectral

- from (5), we have [2]

$$\begin{aligned}\mathcal{P}_x &= \lim_{a\to\infty} \frac{1}{a}\mathcal{E}_{x_a} \\ &= \lim_{a\to\infty} \frac{1}{a}\int_{-\infty}^{\infty} |X_a(f)|^2 df \\ &= \int_{-\infty}^{\infty} \lim_{a\to\infty} \frac{1}{a}|X_a(f)|^2 df \\ &= \int_{-\infty}^{\infty} \mathcal{G}_x(f) df\end{aligned}$$

where $\mathcal{G}_x(f) \triangleq \lim_{a\to\infty} \frac{1}{a}|X_a(f)|^2$ is called the power spectral density (PSD) of x

[2] To interchange the limit and the integral operations, certain technical conditions should hold.

Power: Periodic Signals

- let $x(t)$ be a continuous periodic deterministic signal [3]
- the average power $\mathcal{P}_x$ of x is simply

$$\mathcal{P}_x = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{6}$$

where T is the period of the signal

[3] Energy of x is not finite

PSD : Periodic Signals

- The Fourier Series Expansion of $x(t)$ is given by [4]

$$x(t) = \frac{1}{\sqrt{T}} \sum_{-\infty}^{\infty} c_k e^{jk2\pi t/T} \tag{7}$$

where c_k is given by

$$c_k = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} x(t) e^{-jk2\pi t/T} dt \tag{8}$$

[4] The equality sign symbols used in this slide has a meaning which goes beyond its usual meaning. These subtleties are extraneous to the main scope of the module.

PSD : Periodic Signals

- The PDS of $x(t)$ is given by

$$\mathcal{G}_x(f) \triangleq \frac{1}{T} \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - n/T) \tag{9}$$

Density Spectrum of Discrete Signals

- similar density spectrum definitions are there for discrete signals as well

Density Spectrum of Random Signals

- similar density spectrum definitions are there for random signals as well
- random signals do not directly admit a Fourier transform
- it is possible however to develop a frequency domain representation of the random signals by invoking some advanced concepts

Importance of the Spectral Representation

- signals and systems have both a time-domain and a frequency-domain representation
- pivotal in designs of communication systems
- shows the important spectral components of the signal
- the concept of bandwidth (BW)

Bandwidth of a Signal

- null-to-null bandwidth: the width of the main spectral lobe
- fractional power-containment bandwidth: the frequency interval that contains $(1 - \epsilon)$ of the signal power in positive frequencies
- bounded power-spectral-density bandwidth: the frequency interval such that power density spectrum value is some fraction of the maximum value, e.g., 50dB below the maximum value
- there are more

Bandwidth as a Resource

- BW is very expensive, because it is a resource
- Sri Lanka: BW is controlled by TRC
- there are licenced BW and unicenced BW, e.g.,
  - frequency ranges for use in Wi-Fi: 2.4 GHz, 3.6 GHz, 4.9 GHz, 5 GHz, 5.9 GHz