Introduction to Telecommunications
EN1052

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Lecture 3: Signal and Systems

Relevance ?

- mathematical tools for the analysis and design of digital communication systems
  - signal theory
  - systems theory
  - probability
  - stochastic processes

Discrete Signals and Systems

Discrete-Time Signals

- discrete-time signal: a sequence of real or complex numbers
- denoted by $(x_n)_{n=n_1}^{n_2}$
- when $n_1 = -\infty$ and $n_2 = \infty$, we shall simply write (x_n)

Examples

- discrete-time unit impulse (or unit sample) sequence (δ_n) is defined as

$$\delta_n = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

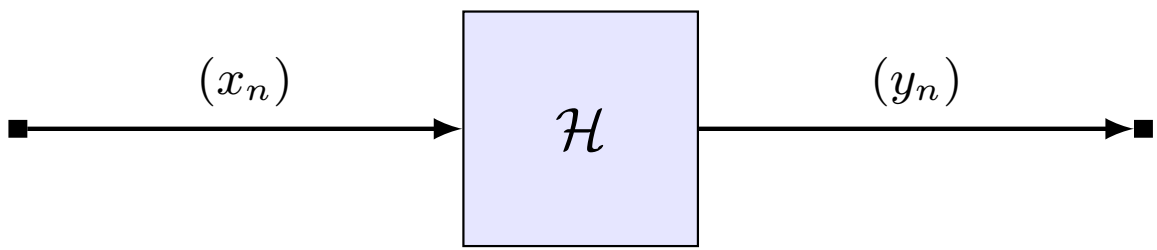
- discrete-time unit step sequence (u_n) is defined as

$$u_n = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- note that $u_n = \sum_{m=-\infty}^{n} \delta_m$

Discrete-Time Systems

- discrete system: a mapping $\mathcal{H}$ of a sequence (x_n) into another sequence (y_n)
- denoted by $y_n = \mathcal{H}[(x_n)]$
- (x_n): input, (y_n): output or response



Examples

- $y_n = x_{n-2}$
- $y_n = x_{n-1} + 2x_n + x_{n+1}$

LTI Systems

Linearity

- a discrete-time system is linear if, for any pair of input signals $(\bar{x}_n)$, $(\tilde{x}_n)$ and for any pair of complex numbers $\bar{a}$, $\tilde{a}$ the following holds:

$$\begin{aligned}\mathcal{H}\left[(\bar{a}\bar{x}_n + \tilde{a}\tilde{x}_n)\right] &= \bar{a}\mathcal{H}\left[(\bar{x}_n)\right] + \tilde{a}\mathcal{H}\left[(\tilde{x}_n)\right] \\ &= \bar{a}\bar{y}_n + \tilde{a}\tilde{y}_n\end{aligned}$$

  where $\bar{y}_n = \mathcal{H}[(\bar{x}_n)]$ and $\tilde{y}_n = \mathcal{H}[(\tilde{x}_n)]$

- i.e., if the system input is a linear combination of two signals, its output is the same linear combination of the two responses

Examples

- the accumulator system $y_n = \sum_{k=-\infty}^{n} x_k$ is linear
- the system $y_n = \log(|x_n| + 1)$ is nonlinear

Time-Invariance

- a discrete-time system is time-invariant if, for any input signals (x_n) the following holds:

$$\mathcal{H}\left[(x_{n-k})\right] = y_{n-k} \quad \text{for all integers } k$$

  where $y_n = \mathcal{H}[(x_n)]$

- i.e., for all delays of an input sequence the output sequence undergoes the same delay

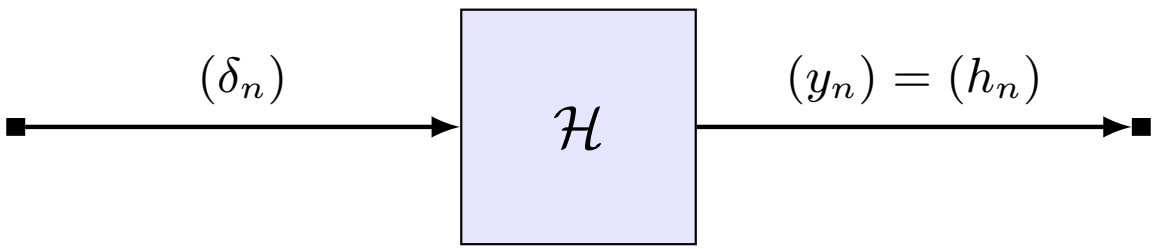
Examples

- the accumulator system $y_n = \sum_{k=-\infty}^{n} x_k$ is time-invariant
- the compressor system $y_n = x_{2n}$ is not time-invariant

Linear Time-Invariant Systems

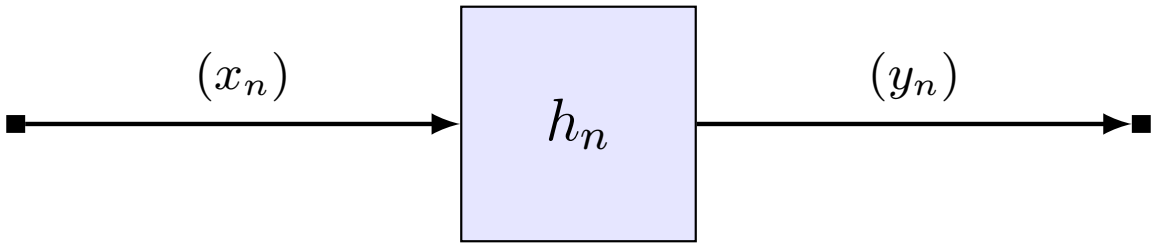
- a discrete-time system which is linear and time-invariant is called a linear time-invariant (LTI) system

Impulse Response of LTI Systems



- the response (h_n) of an LTI system to the input (δ_n) is called the (discrete) impulse response of the system

Response of LTI Systems to (x_n)



- the response (y_n) of an LTI system to any input (x_n) can be characterized by using (h_n)
- in particular, we have

$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} = \sum_{k=-\infty}^{\infty} k_k x_{n-k} = x_n * h_n$$

Discrete-Time Signals and Systems in the Transform Domain

Disc.-Time Signals in Transform Domain [1]

- given a sequence (x_n), the discrete-time Fourier transform (DTFT) of (x_n) is define as

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x_n e^{-jn\theta}$$

- $X(\cdot)$ is a periodic function of θ with period 2π

- the inverse DTFT to compute the elements of (x_n) is given by

$$x_n = \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) e^{jn\theta} d\theta$$

[1] The equality sign symbols used in this slide has a meaning which goes beyond its usual meaning. In addition, we assume that the indicated series is *convergent* and the indicated integral exists, without giving details. All above subtleties are extraneous to the main scope of the module.

Disc.-Time Systems in Transform Domain

- let (h_n) denote the impulse response of an LTI system
- let $H(e^{j\theta})$ be the the DTFT of (h_n)
- $H(e^{j\theta})$ is called the frequency response of the system
- the response of an LTI system with transfer function $H(e^{j\theta})$ to a sequence with Fourier transform $X(e^{j\theta})$ is given by $Y(e^{j\theta}) = H(e^{j\theta})X(e^{j\theta})$

Continuous Signals and Systems

Continuous-Time Signals

- continuous-time signal: real or complex function $x(\cdot)$ of the real variable t (the time)
- denoted by $x(t)$
- time t is assumed to range from $-\infty$ to ∞

Examples

- the continuous-time complex exponential is of the form

$$x(t) = Ce^{at},$$

  where C and a are, in general, complex numbers

- real exponential: $C, a \in \mathbb{R}$
- periodic complex exponential: $x(t) = e^{j\omega_0 t} = e^{j2\pi f_0 t}$, i.e., a is purely imaginary, period ?
- sinusoidal: $x(t) = A\cos(\omega_0 t + \phi) = A\cos(2\pi f_0 t + \phi)$

Examples

- Euler's relation

$$e^{j(2\pi f_0 t+\phi)} = \cos(2\pi f_0 t + \phi) + j\sin(2\pi f_0 t + \phi)$$

  where C and a are, in general, complex numbers

- $A\cos(2\pi f_0 t + \phi)$ in terms of periodic complex exponentials?
- $A\sin(2\pi f_0 t + \phi)$ in terms of periodic complex exponentials?
- fundamental frequency of a sinusoidal: $A\cos(2\pi f_0 t + \phi)$?

Examples

- recall the Euler's relation

$$e^{j(2\pi f_0 t+\phi)} = \cos(2\pi f_0 t + \phi) + j\sin(2\pi f_0 t + \phi)$$

  where C and a are, in general, complex numbers

- $A\cos(2\pi f_0 t + \phi)$ in terms of periodic complex exponentials?
- $A\sin(2\pi f_0 t + \phi)$ in terms of periodic complex exponentials?
- fundamental frequency of a sinusoidal: $A\cos(2\pi f_0 t + \phi)$?

Examples

- reading assignment: read § 1.3.2 and § 1.3.3 of the book Signals and Systems, by Oppenheim, Willsky, and Nawab, second edition

Examples

- unit impulse $\delta(t)$
- roughly speaking, unit impulse $\delta(t)$ is unbounded at $t = 0$ and is an infinitely narrow pulse with area under the graph is 1, i.e.,

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$$

- see page 32 of the Appendix for some properties of $\delta(t)$

Examples

- continuous-time unit step function is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- continuous-time unit impulse, denoted $\delta(t)$ conforms to

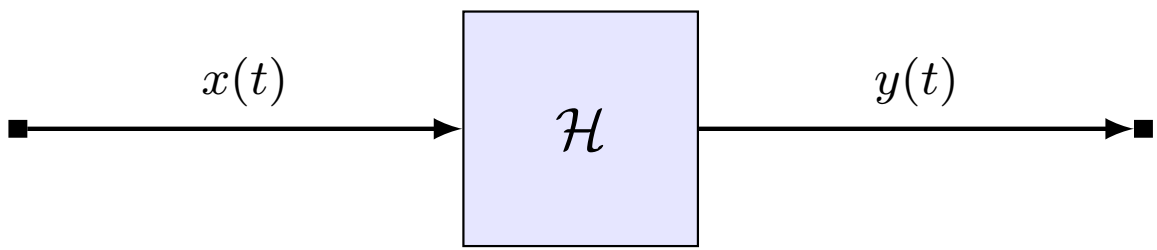
$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

- the unit impulse $\delta(t)$ enables us to give a meaning to the derivative of a function at a discontinuity, i.e.,

$$\delta(t) = \frac{du(t)}{dt}$$

Continuous-Time Systems

- continuous-time system: a mapping $\mathcal{H}$ of a signal $x(t)$ into another signal $y(t)$
- denoted by $y(t) = \mathcal{H}[x(t)]$
- $x(t)$: input, $y(t)$: output or response



Examples

- $y(t) = x(t - 2.1)$
- $y(t) = x(t - 1.5) + 2x(t) + x(t + 2) + 3$

LTI Systems

Linearity

- a continuous-time system is linear if, for any pair of input signals $\bar{x}(t)$, $\tilde{x}(t)$ and for any pair of complex numbers $\bar{a}$, $\tilde{a}$ the following holds:

$$\begin{aligned}\mathcal{H}\left[\bar{a}\bar{x}(t)+\tilde{a}\tilde{x}(t)\right] &= \bar{a}\mathcal{H}\left[\bar{x}(t)\right]+\tilde{a}\mathcal{H}\left[\tilde{x}(t)\right] \\ &= \bar{a}\bar{y}(t)+\tilde{a}\tilde{y}(t)\end{aligned}$$

  where $\bar{y}(t)=\mathcal{H}[\bar{x}(t)]$ and $\tilde{y}(t)=\mathcal{H}[\tilde{x}(t)]$

- i.e., if the system input is a linear combination of two signals, its output is the same linear combination of the two responses

Time-Invariance

- a continuous-time system is time-invariant if, for any input signals $x(t)$ the following holds:

$$\mathcal{H}\left[x(t-\tau)\right] = y(t-\tau) \quad \text{for all } \tau \in \mathbb{R}$$

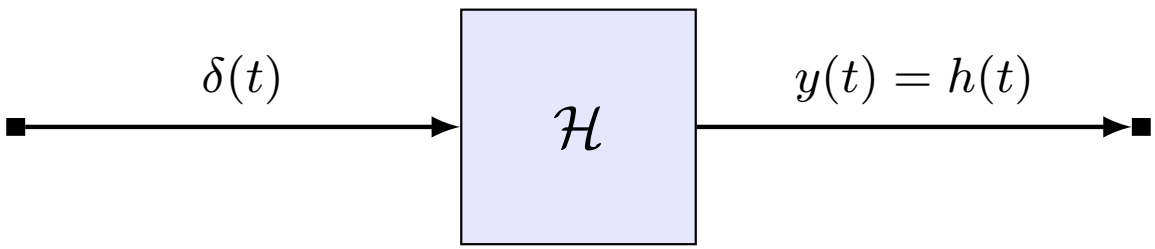
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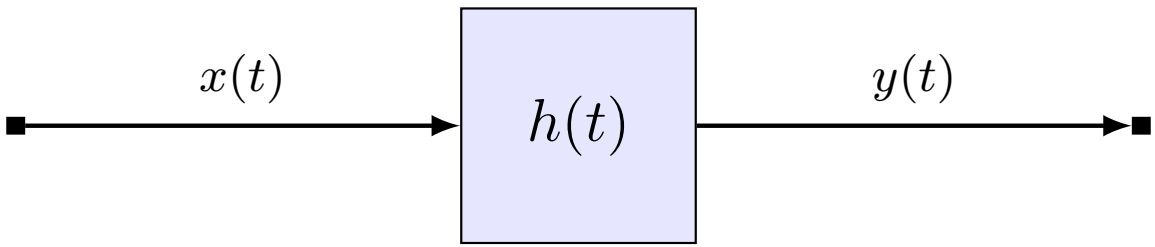
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- the response $h(t)$ of an LTI system to the input $\delta(t)$ is called the (continuous) impulse response of the system

Response of LTI Systems to $x(t)$



- the response $y(t)$ of an LTI system to any input $x(t)$ can be characterized by using $h(t)$
- in particular, we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = x(t) * h(t)$$

Continuous-Time Signals and Systems in the Transform Domain

Cont.-Time Signals in Transform Domain [2]

- given a signal $x(t)$, the Fourier transform (FT) of $x(t)$ is define as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

- the inverse FT to compute the elements of $x(t)$ is given by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

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Cont.-Time Systems in Transform Domain

- let $h(t)$ denote the impulse response of an LTI system
- let $H(f)$ be the the FT of $h(t)$
- $H(f)$ is called the frequency response of the system
- the response of an LTI system with transfer function $H(f)$ to a signal with Fourier transform $X(f)$ is given by $Y(f) = H(f)X(f)$

Appendix

Some Properties of $\delta(t)$

- sampling or shifting property:

$$\int_{-\infty}^{\infty} g(t)\delta(t-T)dt = g(T)$$

  assuming $g(t)$ is continuous at $t = T$

- convolution (assuming $g(t)$ is continuous)

$$g(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau) \underbrace{g(t-\tau)}_{f(\tau)} d\tau = \int_{-\infty}^{\infty} f(\tau)\delta(\tau - 0)d\tau$$

$$= \int_{-\infty}^{\infty} f(0)\delta(\tau - 0)d\tau = g(t)\int_{-\infty}^{\infty} \delta(\tau - 0)d\tau = g(t)$$