Mathematical Optimization
YTU 2019

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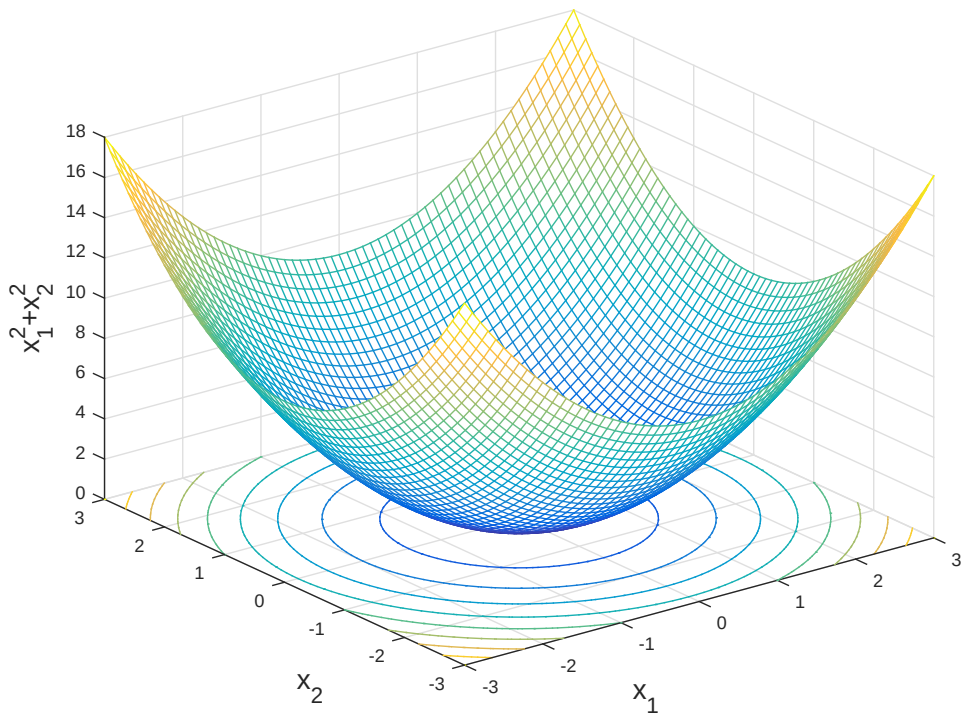
August 2019

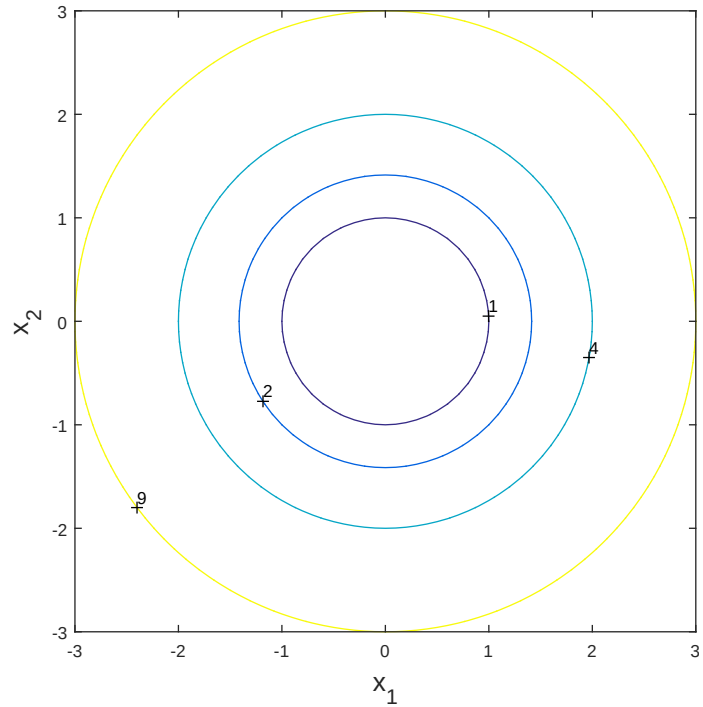
Lecture 9: Math Review: Part 8

Level Sets

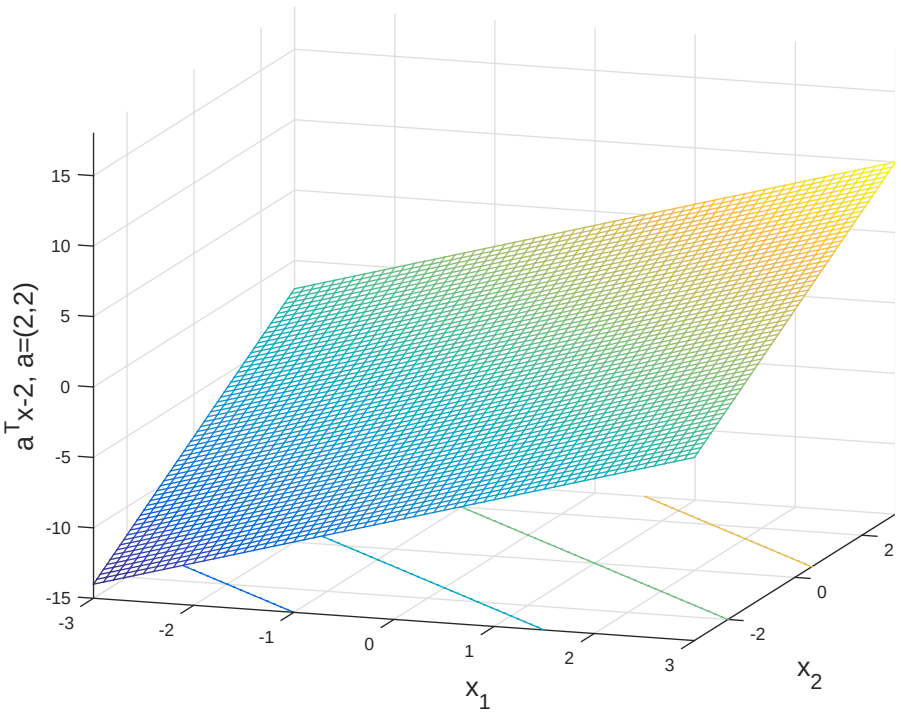
- the level set of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at level c is the set of points
$$\mathcal{S} = \{x \mid f(x) = c\}$$
- when
  - $n = 1$, $\mathcal{S} \subseteq \mathbb{R}$
  - $n = 2$, $\mathcal{S} \subseteq \mathbb{R}^2$
  - $n = 3$, $\mathcal{S} \subseteq \mathbb{R}^3$
- graphically illustrate the idea
- check Example 5.2 of ref-1

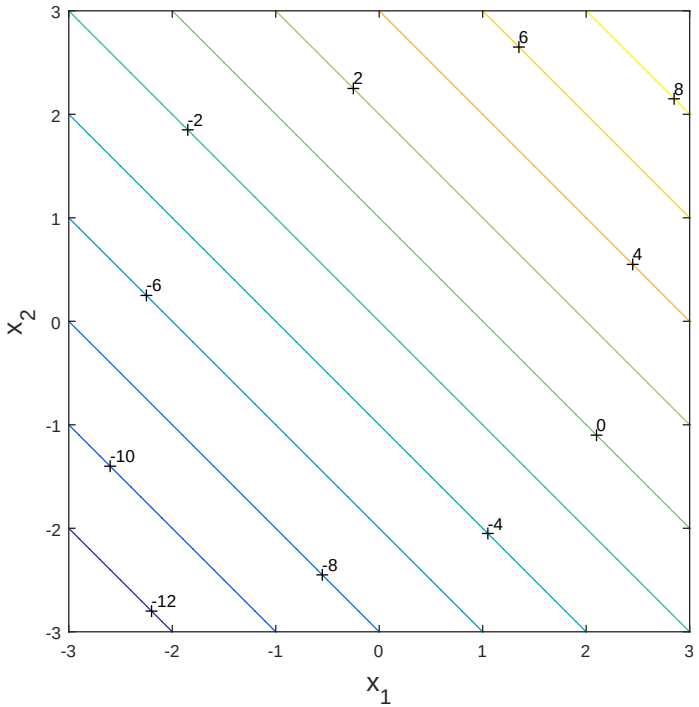
Level Sets of $f(x) = ||x||^2$





Level Sets of $f(x) = a^T x - 2,\ a = (2, 2)$





Level Sets and Gradient

- suppose the considered functions are differentiable
- level sets of a function f and the gradient of $\nabla f(x)$ of f are inherently related
- let $\mathcal{S}$ be the level set of f at c , i.e.,

$$\mathcal{S} = \{x \mid f(x) = c\}$$

- let x_0 be a point in $\mathcal{S}$, i.e., $x_0 \in \mathcal{S}$
- Theorem 5.7 of ref-1: the gradient $\nabla f(x_0)$ is orthogonal to the tangent vector to an arbitrary smooth curve on the level set $\mathcal{S}$ that passes through x_0

Level Sets and Gradient

- illustrate the idea

Sublevel Sets

- the sublevel set of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at level c is the set of points
$$\mathcal{S} = \{x \mid f(x) \leq c\}$$
- again when
  - $n = 1$, $\mathcal{S} \subseteq \mathbb{R}$
  - $n = 2$, $\mathcal{S} \subseteq \mathbb{R}^2$
  - $n = 3$, $\mathcal{S} \subseteq \mathbb{R}^3$
- graphically illustrate the idea

Other Interpretations for $\nabla f(x)$

- let us consider the level set of f at a level $c = f(x_0)$, i.e.,

$$\mathcal{S}_{f(x_0)} = \{x \mid f(x) = f(x_0)\}$$

- note that $x_0 \in \mathcal{S}_{f(x_0)}$
- $\nabla f(x_0)$, if nonzero, defines the tangent plane to $\mathcal{S}_{f(x_0)}$ at x_0
- $\nabla f(x_0)$, if nonzero, defines the hyperplane that supports $\mathcal{S}_{f(x_0)}$ at x_0

Other Interpretations for $\nabla f(x)$

- in particular, the tangent plane (or hyperplane) to $\mathcal{S}_{f(x_0)}$ at x_0 is given by the set

$$\mathcal{S}_{\texttt{tangent}} = \{x \mid \nabla f(x_0)^T (x - x_0) = 0\}$$

- verify that $\mathcal{S}_{\texttt{tangent}} = \mathcal{S}_{f_1(x_0)}$, where f_1 is the first order approximation of f at x_0

- $\nabla f(x_0)$ is the direction of maximum rate of increase of f at x_0

- check the Theorem 5.7, Ref-1 and Figure 5.4 of Ref-1

Other Interpretations for $\nabla f(x)$

- $\nabla f(x_0)$ can also be interpreted by using the graph of f
- let us create the vector a by concatenating gradient $\nabla f(x_0)$ together with -1, i.e.,

$$a = \begin{bmatrix} \nabla f(x_0) \\ -1 \end{bmatrix}$$

- a defines the hyperplane that supports graph of f at $(x_0, f(x_0))$
- graphically illustrate the idea

Can We Prove the Aforementioned Result?

- sketch of the proof: we are given $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
  - define the function $g : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ as

$$g(x, y) = f(x) - y$$

  - determine the level set of $g(x, y)$ at level 0, i.e.,

$$\mathcal{S} = \{(x, y) \mid g(x, y) = 0\} = \{(x, y) \mid y = f(x)\} \qquad (1)$$

  - compute the gradient of g at (x_0, y_0), i.e., $\nabla g(x_0, y_0)$

$$\nabla g(x_0, y_0) = \nabla \big(f(x) - y\big) \Big|_{(x,y)=(x_0,y_0)} = \begin{bmatrix} \nabla f(x_0) \\ -1 \end{bmatrix}$$

  - recall that $\nabla g(x_0, y_0)$ defines the hyperplane that support $\mathcal{S}$ [see (1)] at (x_0, y_0), what this means?

Level Sets and Gradient

- graphically show the normal vector of the graphs of
  - $2x_1 + 3$ at $(1, 5)$
  - $\log(3x_1)$ at $(2, \log(6))$
  - $\exp(2x_1)$ at $(2, \exp(4))$
  - $2x_1 + 5x_2$ at $(1, 1, 7)$
  - $x_1^2 + x_2^2$ at $(1, 1, 2)$