Mathematical Optimization
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Lecture 7: Math Review: Part 6

Linear Equations

- suppose we are given m equations in n unknowns of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- define $a_i = (a_{1i}, a_{2i}, \ldots, a_{mi}) \in \mathbb{R}^m$, $i = 1, 2, \ldots, n$

- then the above set of equations can also be given by

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b,$$

where $b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$

Linear Equations

- we can further compactly represent the above equations as follows

$$[a_1 \ a_2 \ \cdots a_n] \ x = b,$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

- if we let $A = [a_1 \ a_2 \ \cdots a_n]$, we get the system of equations

$$A \ x = b$$

- it is recommended to be familiar with block matrix operations!

Solution of a Linear System

- there can be
  - many solutions
  - unique solution
  - no solution
- exemplify the above 3 case !

Solution of a Linear System

- let $[A\ b]$ denote the matrix by concatenating $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$
- The linear system $Ax = b$ has a solution if and only if

$$\text{rank } A = \text{rank } [A\ b]$$

- see Theorem 2.1 of ref-1 for a proof!
- the above suggests that

$$\dim\big(\text{span}\{a_1, a_2, \ldots, a_n\}\big) = \dim\big(\text{span}\{a_1, a_2, \ldots, a_n, b\}\big)$$

- i.e., b can be written as a linear combination of the vectors $a_1, a_2, \ldots, a_n$

Solution of a Linear System

- More specifically, $Ax = b$, $A \in \mathbb{R}^{m \times n}$
- has no solution:

$$\text{rank}A < \text{rank}[A\ b]$$

- has solution:
  - unique: rank A = rank $[A\ b] = n$
  - many: rank A = rank $[A\ b] < n$
- graphically illustrate the 3 cases above