Mathematical Optimization
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Lecture 6: Math Review: Part 5

Matrices

- a matrix is a rectangular array of numbers written between rectangular brackets, e.g.,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

- in general a matrix can have m rows and n columns, i.e., $A \in \mathbb{R}^{m\times n}$
- transpose of A is denoted $A^T \in \mathbb{R}^{n\times m}$
- note that $\mathbb{R}^{m\times n}$ is the set of real $m \times n$ matrices
- $\mathbb{R}^{n\times 1} = \mathbb{R}^n$ based on our notations

Matrices

- some useful compact representations of $A \in \mathbb{R}^{m \times n}$
- represent in terms of columns:

$$A = [a_1 \; a_2 \; \cdots a_n],$$

where $a_i \in \mathbb{R}^m$, $i = 1, 2, \ldots, n$

- represent in terms of rows:

$$A = \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_m^T \end{bmatrix},$$

where $\tilde{a}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$

Rank of a Matrix

- let the matrix $A \in \mathbb{R}^{m \times n}$ is given by

$$A = [a_1 \ a_2 \ \cdots a_n],$$

  where $a_i \in \mathbb{R}^m$, $i = 1, 2, \ldots, n$

- rank A = maximum number of linearly independent columns of A
- rank A = maximum number of linearly independent rows of A
- thus, rank $A \leq \min(m, n)$

Rank of a Matrix

- exercise:
  - determine the rank of following matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 0 \end{bmatrix}$$

  - Matlab can be used to easily determine the rank of a matrix, check
  - many algorithms to determine rank, e.g., GramSchmidt procedure

Rank of a Matrix

- rank of a matrix A is invariant under
  - multiplication of the columns of A by a nonzero scalar
  - interchange of columns
  - addition to a given column a linear combination of others
- check proposition 2.3 of Ref-1

Nullspace of a Matrix

- the nullspace of $A \in \mathbb{R}^{m \times n}$ is given by

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

- $\mathcal{N}(A)$ is the set of vectors mapped to 0 when multiplied by A
- $\mathcal{N}(A)$ is the set of vectors orthogonal to rows of A
- $\mathcal{N}(A)$ gives ambiguity in x given $y = Ax$
  - if $y = Ax$ and $z \in \mathcal{N}(A)$, then $y = A(x + z)$

Zero Nullspace

- 0 is the only element of $\mathcal{N}(A)$, i.e., $\mathcal{N}(A) = \{0\}$
- $\mathcal{N}(A) = \{0\} \iff$
  - x can always be uniquely determined from $y = Ax$
  - A is one-to-one, i.e., different x's map to different y's
  - columns of A are independent (hence, a basis for span$\{a_1, \ldots, a_n\}$)
  - det$(A^T A) \neq 0$
- all of the above equivalences can be proved, some obvious, some not

Range of a Matrix

- the range of $A \in \mathbb{R}^{m \times n}$ is given by

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

- note that $\mathcal{R}(A) \subseteq \mathbb{R}^m$
- $\mathcal{R}(A)$ is the set of vectors that can be hit by mapping $y = Ax$
- $\mathcal{R}(A) = \text{span}\{a_1, \ldots, a_n\}$
- $\mathcal{R}(A)$ is the set of vectors y for which $Ax = y$ has a solution

Onto Matrix

- $\mathcal{R}(A)$ is identical to $\mathbb{R}^m$, i.e., $\mathcal{R}(A) = \mathbb{R}^m$, A is called onto
- $\mathcal{R}(A) = \mathbb{R}^m \iff$
  - $Ax = y$ can be solved in x for any y
  - columns of A spans $\mathbb{R}^m$
  - rows of A are independent
  - $\det(AA^T) \neq 0$
  - $\mathcal{N}(A^T) = \{0\}$
- all of the above equivalences can be proved, some obvious, some not

Square Matrices

- number of rows is equal to the number of columns
- for such a square matrices A we define the determinant of A, denoted $|A|$ or $\det A$
- if $|A| \neq 0$, we can also define the inverse of A, denoted A^{-1}
- read pages 14-15 of ref-1, (more details chapter 6 of Ref-2)
- special square matrices: e.g., identity matrix, diagonal matrices