Mathematical Optimization
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Lecture 5: Math Review: Part 4

Euclidean Inner Product

- for any $x, y \in \mathbb{R}^n$, we define the Euclidean inner product as

$$x^T y$$

- note that

$$x^T y = y^T x = \sum_{i=1}^{n} x_i y_i$$

- the inner product is a real-valued function f, i.e.,

$$f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

Properties of Inner Product

- positivity, i.e.,

$$x^T x \geq 0 \quad \text{and} \quad x^T x = 0 \iff x = 0$$

- symmetry, i.e.,

$$x^T y = y^T x$$

- additivity, i.e.,

$$(x + y)^T z = x^T z + y^T z$$

- homogeneity, i.e.,

$$(rx)^T y = r\ x^T y \quad \text{for all } r \in \mathbb{R}$$

Euclidean Norm of a Vector

- Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as

$$||x|| = \sqrt{x^T x}$$

- let $x = (x_1, x_2) \in \mathbb{R}^2$, then

$$||x|| = \sqrt{x_1^2 + x_2^2}$$

- let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

- graphically illustrate norm of a vector in $\mathbb{R}^2$ and in $\mathbb{R}^3$

Properties of Euclidean Norm

- positivity, i.e.,

$$||x|| \geq 0 \quad \text{and} \quad ||x|| = 0 \iff x = 0$$

- homogeneity, i.e.,

$$||rx|| = |r| \; ||x|| \quad \text{for all } r \in \mathbb{R}$$

- triangular inequality , i.e., for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

$$||x + y|| \leq ||x|| + ||y||$$

Orthogonal Vectors

- any two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are orthogonal [perpendicular] whenever the angle between those two vectors are 90°
- essence of orthogonality in $\mathbb{R}^2$ or $\mathbb{R}^3$ is described by Pythagorean Theorem, i.e.,

$$
\begin{aligned}
||u||^2 + ||v||^2 &= ||u - v||^2 \\
u^T u + v^T v &= (u - v)^T (u - v) \\
u^T u + v^T v &= u^T u - u^T v - v^T u + v^T v \\
\therefore \; 2\, u^T v &= 0 \\
u^T v &= 0, \text{ i.e., the Euclidean inner product is zero}
\end{aligned}
$$

Orthogonal Vectors

- similarly, two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ are said to be orthogonal to each other whenever

$$u^T v = 0$$

- of course, visual concept of right angle is not at our disposal in higher Euclidean space

Cauchy-Schwarz Inequality

- for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we have

$$|x^T y| \leq ||x||\ ||y||$$

  and equality holds if and only if $x = \alpha y$ for some $\alpha \in \mathbb{R}$

- see Theorem 2.3 of ref-1, proof is also available there

- exercise: interpret Cauchy-Schwarz inequality by using a figure

Triangular Inequality

- exercise: prove the triangular inequality by using Cauchy-Schwarz
- hint: start with $||x + y||^2$ and proceed..
- what if when x and y are orthogonal? Pythagorean for $\mathbb{R}^n$