Mathematical Optimization
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Lecture 13: Convex Optimization Problems

Recall from Lecture 10

- standard formulation of an optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & g_i(x) = 0, \ i = 1, \ldots, p \\ & f_i(x) \leq 0, \ i = 1, \ldots, m \end{array} \tag{1}$$

- decision variable: x
- objective function: $f_0(x)$
- equality constraint: $g_i(x) = 0, \ i = 1, \ldots, p$
- inequality constraint: $f_i(x) \leq 0, \ i = 1, \ldots, m$

Locally Optimal Vs Globally Optimal

- we say $x^\star$ is a globally optimal point, whenever $x^\star \in \mathcal{F}$ and $f_0(x^\star) = p^\star$, where $p^\star = \inf\{f_0(x) \mid x \in \mathcal{F}\}$
- define the set

$$\mathcal{X}_{\text{opt}} = \{x \in \mathcal{F} \mid f_0(x) = p^\star\}$$

- if there exist a globally optimal point for problem (3), we say the optimal value is attained (p 128, ref-3)
- if $\mathcal{X}_{\text{opt}}$ is empty, we say the optimal value is not attained (p 128, ref-3)

Locally Optimal Vs Globally Optimal

- we say x_0 is a locally optimal point, whenever $x_0 \in \mathcal{F}$ and

  $f_0(x_0) = \inf\{f_0(x) \mid x \in \mathcal{F}, ||x - x_0|| \leq R\}$ for some $R > 0$

- i.e., $x_0 \in \mathcal{F}$ minimizes f_0 over some ball around x_0
  (p 128, ref-3)

- the word 'optimal' is used for 'globally optimal' here onwards

Convex Optimization Problems

- we say a problem in standard form is convex whenever:
  - the objective function f_0 is convex AND
  - the inequality constraint functions $f_i(x), i = 1, \ldots, m$ are convex AND
  - the equality constraint functions $g_i(x),\ i = 1, \ldots, p$ are affine

Convex Optimization Problems

- i.e., the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x - b_i = 0,\ i = 1, \ldots, p \\ & f_i(x) \leq 0,\ i = 1, \ldots, m \end{array}$$

  is convex provided the functions $f_i,\ i = 0, \ldots, m$ are convex

Feasible Set of a Convex Problem

- it can easily be shown that the feasible set of a convex optimization problem is also convex
  - i.e., $\mathcal{F}$ is convex
- in other words, in the case of convex optimization problems, we are minimizing a convex function over a convex set
- geometrically interpret the idea with level curves and $\mathcal{F}$

Importance of Convexity

- a fundamental property of convex optimization problems:
  - any locally optimal point is also globally optimal
- discuss the local optimal and global optimal points (p 128, ref-3)
- optimality criterions can be specified (§ 4.2.3, ref-3)

Importance of Convexity

- by using the optimality criterions, efficient algorithms can be designed to find the optimal point
- e.g.,
  - unconstrained optimization: gradient descent algorithm (§ 9.3, ref-3), steepest descent algorithm (§ 9.4, ref-3), Newton's method (§ 9.5, ref-3)
  - equality constrained optimization: Newton's method with equality constraints (§ 10.2, ref-3)
  - constrained optimization: interior-point method (§ 11, ref-3)
  - simplex algorithm: a particular one to solve only linear programs (LP) [1]

[1] LPs are discussed soon.

Importance of Convexity

- the algorithms aforementioned are gracefully implemented in many software packages
- a popular one is
  - `CVX`, matlab software for convex optimization
- for a complete list, you may refer to the stanford website, `http://web.stanford.edu/~boyd/software.html`

Linear Programming

- when the objective and constraint functions are all affine, the problem is called a linear program (LP)

- a general linear program has the form

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b, \end{array} \tag{2}$$

  where $G \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{p \times n}$

- geometrically interpret, what is an LP (Figure 4.4, p. 147, ref-3)

Linear Programming

- e.g., diet problem 

Quadratic Optimization Problems

- when the objective the objective function is (convex) quadratic and the constraint functions are affine, the problem is called a quadratic program (QP)

- a general QP has the form

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b, \end{array} \tag{3}$$

  where P is symmetric positive semidefinite, $G \in \mathbb{R}^{m \times n}$, and $A \in \mathbb{R}^{p \times n}$

- geometrically interpret, what is a QP (Figure 4.5, p. 153, ref-3)

Quadratic Programming

- e.g., least-squares and constrained least-squares