Mathematical Optimization
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Lecture 12: Convex Functions, § 3, ref-3

Definition, Convexity of f

- a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be convex whenever $\texttt{dom}\ f$ is a convex set and if for all $x, y \in \texttt{dom}\ f$, and θ with $0 \leq \theta \leq 1$, we have

$$f\big(\theta x + (1-\theta)y\big) \leq \theta f(x) + (1-\theta) f(y) \tag{1}$$

- geometrically, this inequality means that the line segment between $(x, f(x))$ and $(y, f(y))$ for all x, y, lies above the graph of f

- graphically illustrate the idea

Definition, Strict Convexity of f

- a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be strictly convex whenever $\mathtt{dom}\ f$ is a convex set and if for all $x, y \in \mathtt{dom}\ f, x \neq y$, and θ with $0 \leq \theta \leq 1$, we have

$$f\big(\theta x + (1-\theta)y\big) < \theta f(x) + (1-\theta) f(y)$$

- geometrically, this inequality means that the line segment between $(x, f(x))$ and $(y, f(y))$ for all distinct x, y, lies strictly above the graph of f

- graphically illustrate the idea

Definition, Concave f

- a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be concave whenever $-f$ is convex
- a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be strictly concave whenever $-f$ is strictly convex
- graphically illustrate the idea

Examples on $\mathbb{R}$

- linear functions: function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = ax, \quad a \in \mathbb{R}, \text{ is convex/concave on } \mathbb{R}$$

- affine functions: function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = ax + b, \quad a, b \in \mathbb{R}, \text{ is convex/concave on } \mathbb{R}$$

- exponential: function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = e^{ax}, \quad a \in \mathbb{R}, \text{ is convex on } \mathbb{R}$$

Examples on $\mathbb{R}$

- powers (positive): function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = x^a, \text{ is convex on } \mathbb{R}_+ \text{ when } a \geq 1$$

- powers (negative): function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = x^a, \text{ is convex on } \mathbb{R}_{++} \text{ when } a \leq 0$$

- powers: function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = x^a, \text{ is concave on } \mathbb{R}_{++} \text{ when } a \in [0, 1]$$

- powers of absolute value: function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = |x|^p, \text{ is convex on } \mathbb{R} \text{ for } p \geq 1$$

Examples on $\mathbb{R}$

- logarithm: function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = \log(x), \text{ is concave on } \mathbb{R}_{++}$$

- negative entropy: function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form:

$$f(x) = x \log(x), \text{ is convex on } \mathbb{R}_{+}$$

Examples on $\mathbb{R}^n$

- linear functions: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = a^T x, \quad a \in \mathbb{R}^n, \text{ is convex/concave on } \mathbb{R}^n$$

- affine functions in $\mathbb{R}^n$: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = a^T x + b, \quad a, b \in \mathbb{R}^n, \text{ is convex/concave on } \mathbb{R}^n$$

- affine functions in $\mathbb{R}^{m \times n}$: function $f : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}$ of the form:

$$f(X) = \mathbf{tr}(A^{\mathsf{T}} X) + b, \text{ is convex/concave on } \mathbb{R}^n$$

note: $\mathbf{tr}(A^{\mathsf{T}} X) + b = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} X_{ij} + b$

Examples on $\mathbb{R}^n$

- Euclidean norm square: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = x^T x, \text{ is convex on } \mathbb{R}^n$$

- quadratic form: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = x^T P x + q^T x + r, \text{ is convex on } \mathbb{R}^n, \text{ where}$$

P is symmetric $n \times n$ and positive semidefinite, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}$

Examples on $\mathbb{R}^n$

- Euclidean norm (ℓ_2 norm): function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = ||x|| = ||x||_2, \text{ is convex on } \mathbb{R}^n$$

- ℓ_1 norm: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = ||x||_1 = \sum_{i=1}^{n} |x_i|, \text{ is convex on } \mathbb{R}^n$$

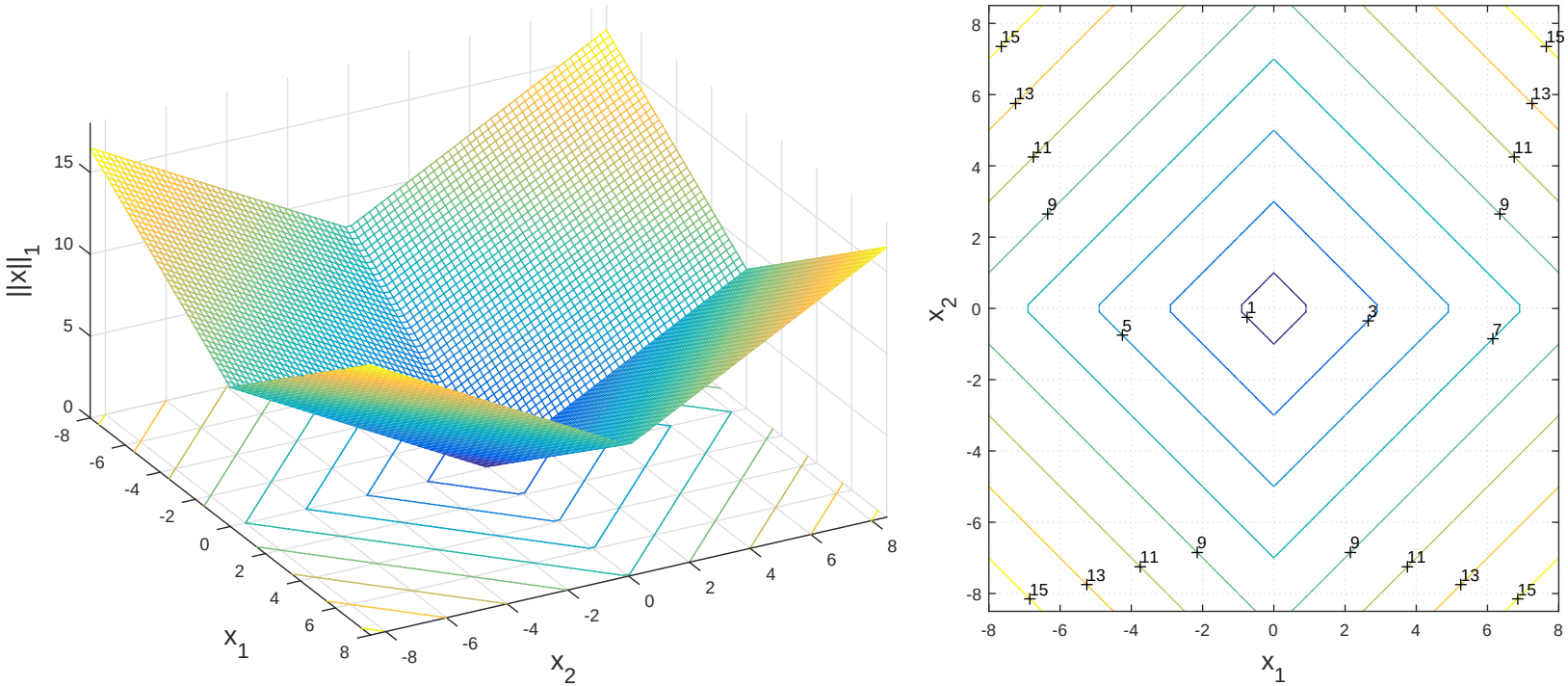
- ℓ_∞ norm: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = ||x||_\infty = \max_{i \in \{1,\ldots,n\}} |x_i| = \max_i |x_i|, \text{ is convex on } \mathbb{R}^n$$

- ℓ_p norm: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

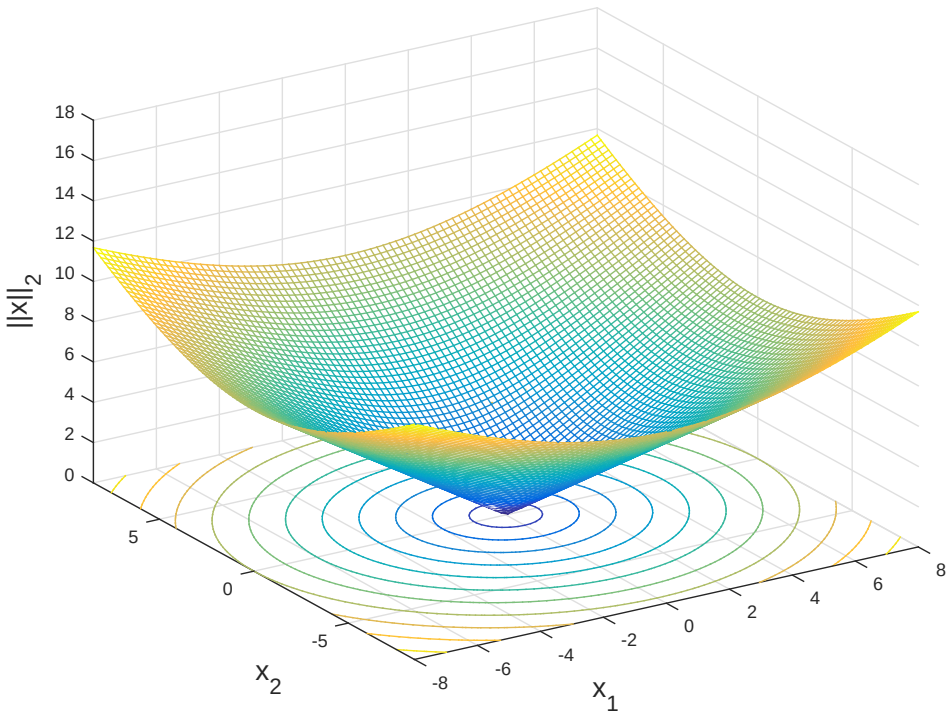
$$f(x) = ||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \ p \geq 1 \text{ is convex on } \mathbb{R}^n$$

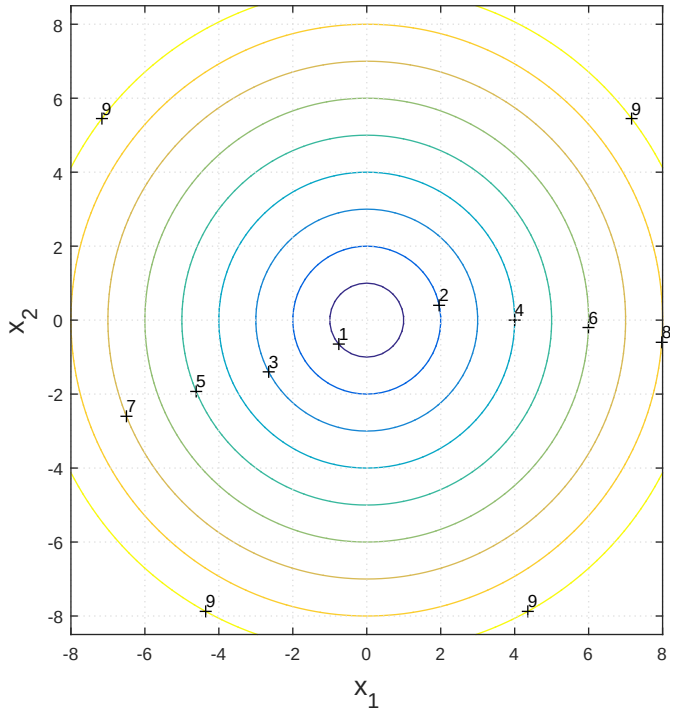
Examples on $\mathbb{R}^n$



- graph of $||x||_1$ and level sets of $||x||_1$

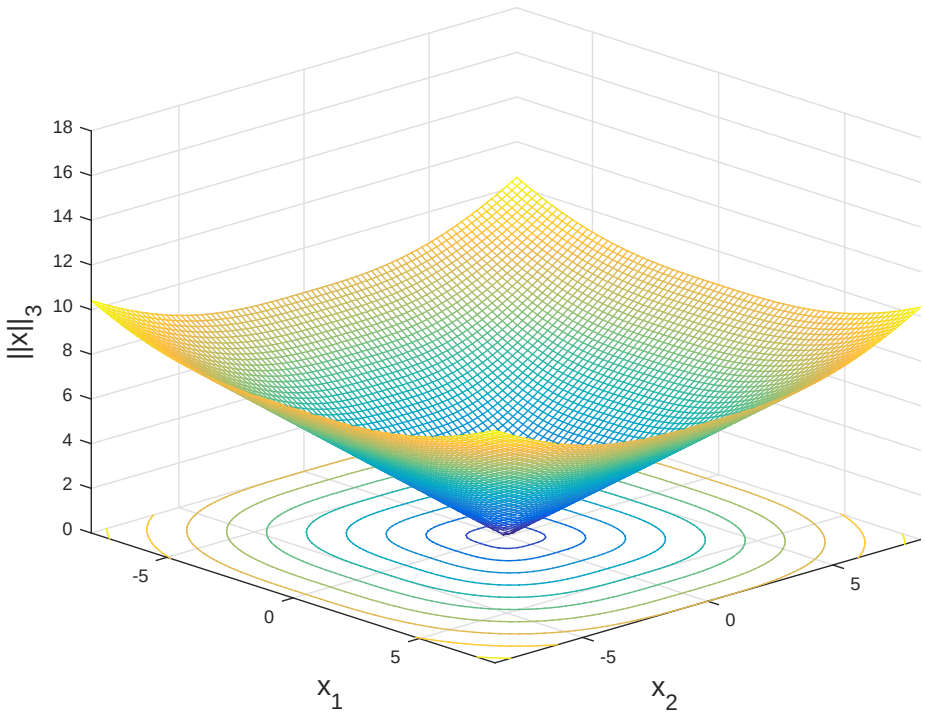
Examples on $\mathbb{R}^n$

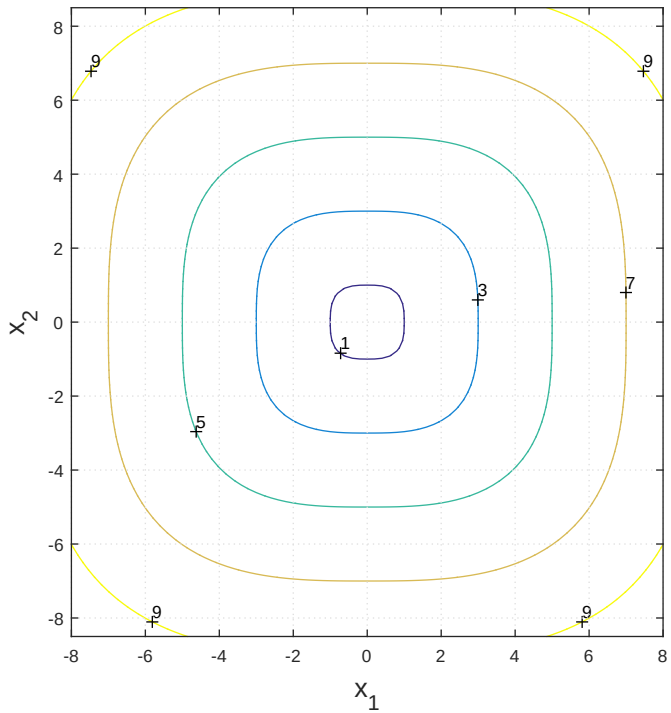




- graph of $||x||_2$ and level sets of $||x||_2$

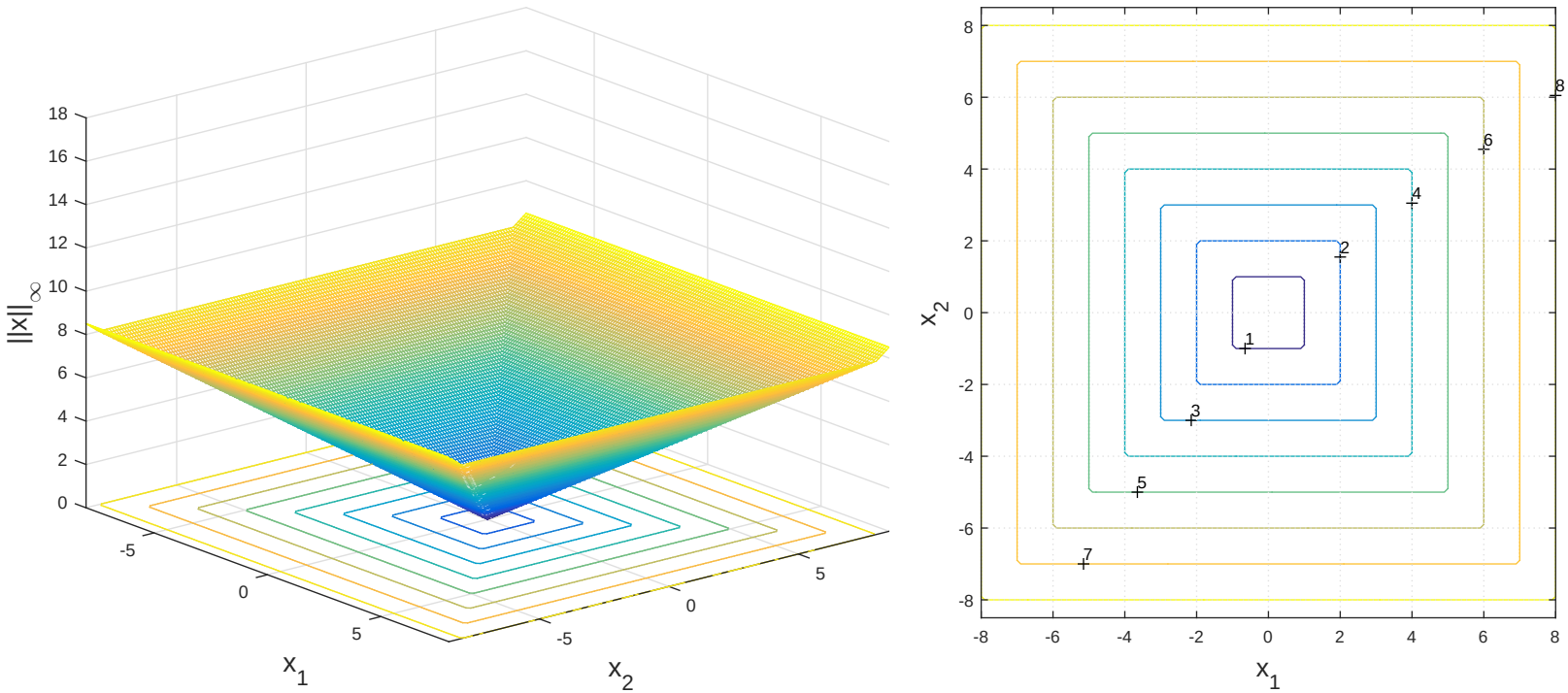
Examples on $\mathbb{R}^n$





- graph of $||x||_p$ and level sets of $||x||_p$ with $p = 3$

Examples on $\mathbb{R}^n$

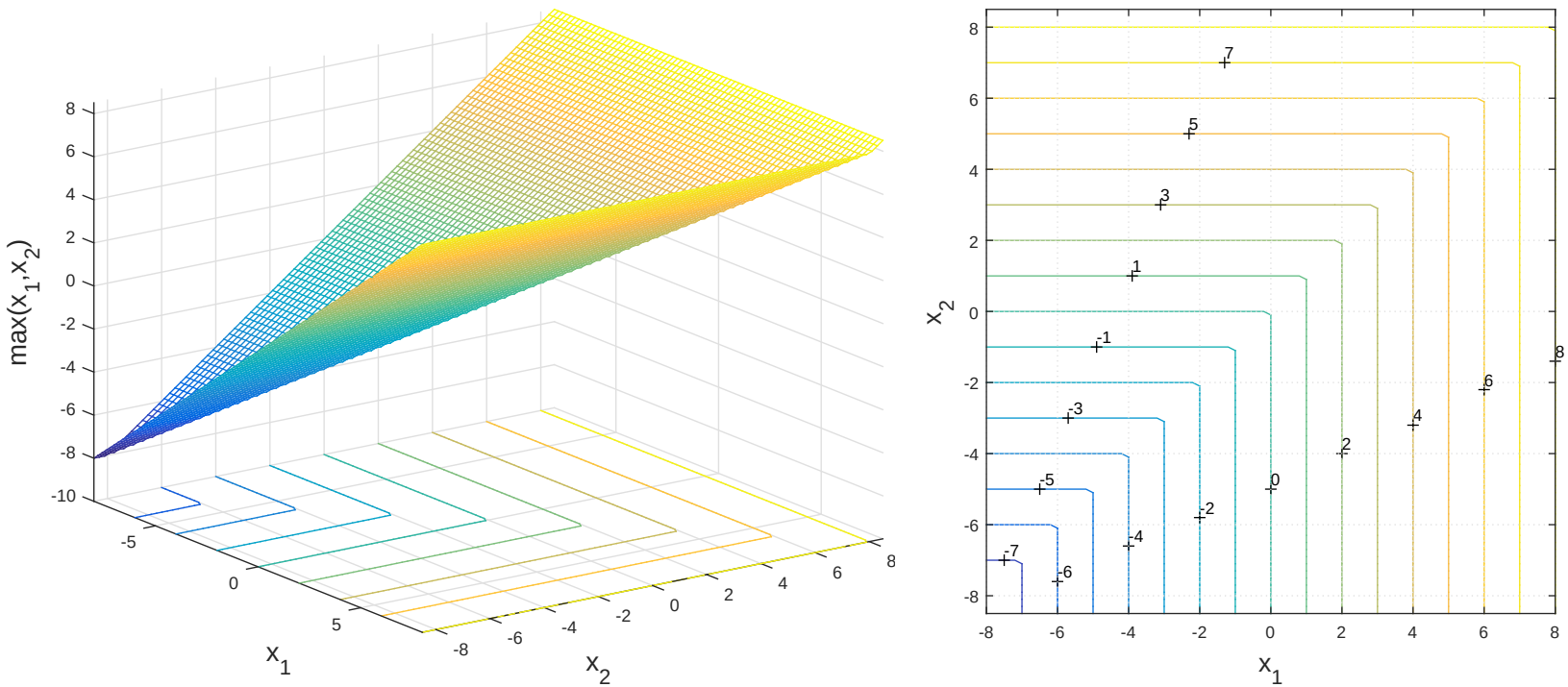


- graph of $||x||_\infty$ and level sets of $||x||_\infty$

Examples on $\mathbb{R}^n$

- max function: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = \max\{x_1, \ldots, x_n\}, \text{ is convex on } \mathbb{R}^n$$

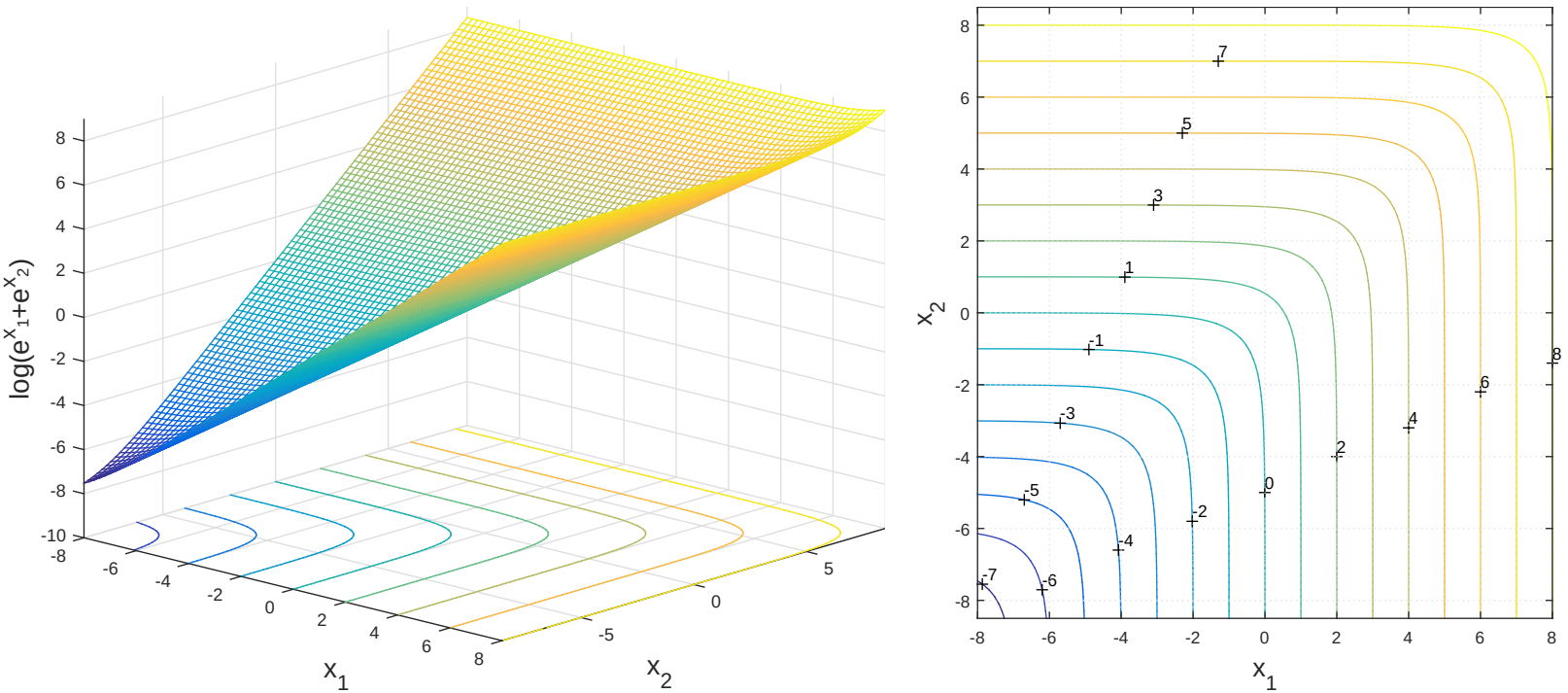


- graph of $\max\{x_1, x_2\}$ and level sets of $\max\{x_1, x_2\}$

Examples on $\mathbb{R}^n$

- log-sum-exp: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = \log\left(e^{x_1} + \cdots + e^{x_n}\right), \text{ is convex on } \mathbb{R}^n$$

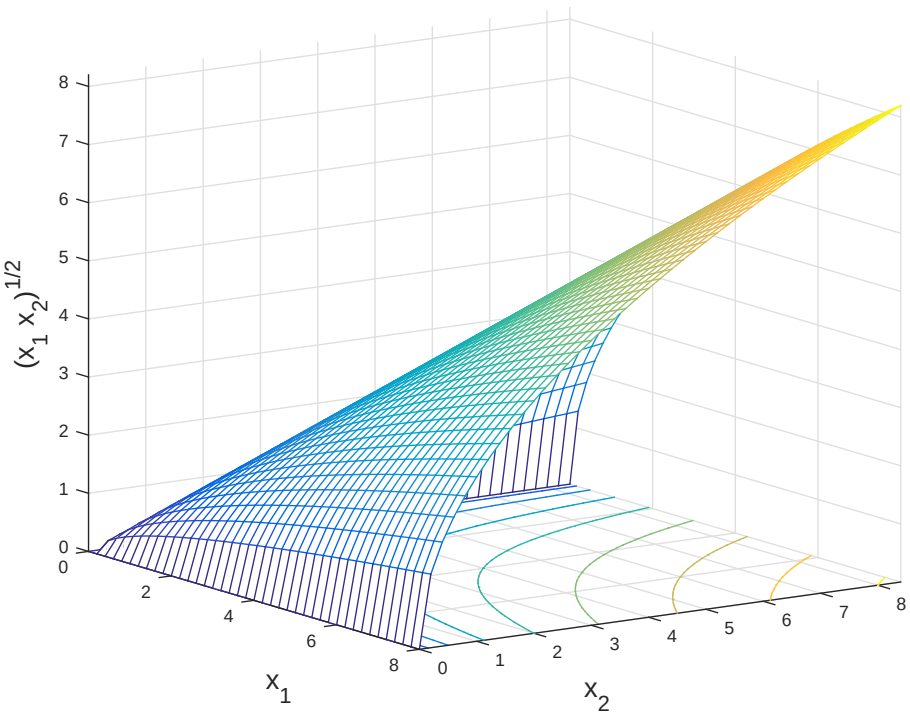


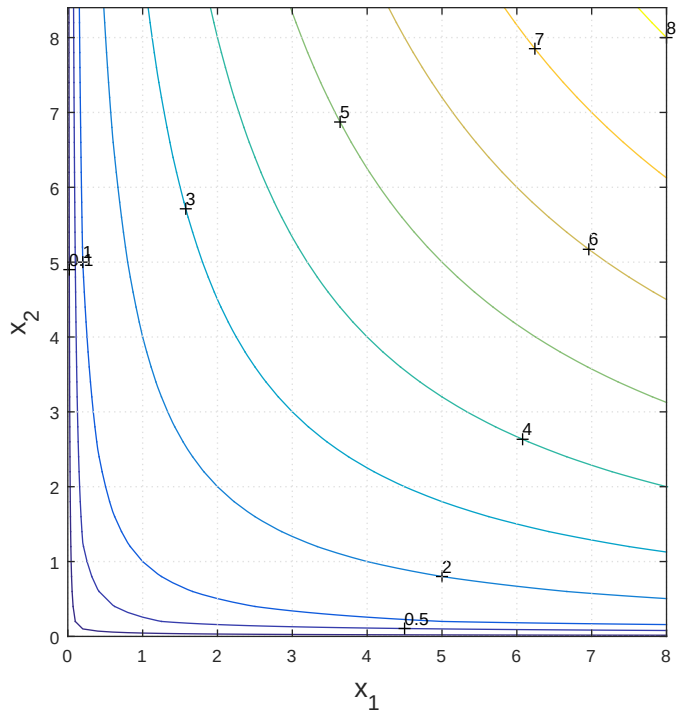
- graph of $\log\left(e^{x_1} + e^{x_2}\right)$ and level sets of $\log\left(e^{x_1} + e^{x_2}\right)$

Examples on $\mathbb{R}^n$

- geometric mean: function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form:

$$f(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \text{ is concave on } \mathbf{dom} f = \mathbb{R}^n_{++}$$





- graph of $(x_1 x_2)^{1/2}$ and level sets of $(x_1 x_2)^{1/2}$

Examples on $\mathbb{R}^n$

- there are many others
- for details you may refer to § 3.1.5 of ref-3

First-Order Conditions, § 3.1.3 of ref-3

- suppose f is differentiable, i.e., the gradient ∇f exists at each point in $\mathbf{dom}\ f$, which is open
- then f is convex if and only if $\mathbf{dom}\ f$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^{\mathsf{T}}(y - x)$$

  for all $x, y \in \mathbf{dom}\ f$
- local information (gradient) leads to global information (convexity)
- f is strictly convex if and only if $\mathbf{dom}\ f$ is convex and

$$f(y) > f(x) + \nabla f(x)^{\mathsf{T}}(y - x)$$

  for all $x, y \in \mathbf{dom}\ f, x \neq y$

Second-Order Conditions, § 3.1.3 of ref-3

- suppose f is twice differentiable, i.e., the Hessian $\nabla^2 f(x)$ and the gradient ∇f exists at each point in $\mathbf{dom}\ f$, which is open
- then f is convex if and only if $\mathbf{dom}\ f$ is convex and its Hessian is positive definite, i.e.,

$$\text{for all } x \in \mathbf{dom}\ f,\ \nabla^2 f(x) \succeq 0$$

- geometrically: graph of f has positive (upward) curvature at each point x in the domain of f
- if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\ f$, then f is strictly convex
  - converse is not true in general

Some Examples

- quadratic function: $f(x) = (1/2)x^{\mathsf{T}}Px + q^{\mathsf{T}}x + r$, where $P \in S^n$

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

f is convex $\iff$ $P \succeq 0$, moreover f is concave $\iff$ $P \preceq 0$

- quadratic-over-linear: $f(x, y) = x^2/y$
  - what is $\nabla^2 f(x)$?
  - is f convex?

Operations Preserving Convexity

- nonnegative weighted sum
- composition with affine function
- pointwise maximum and supremum
- composition
- minimization
- perspective

nonnegative weighted sum

- f is convex $\implies$ αf is convex for any $\alpha \geq 0$
- f_1, f_2 is convex $\implies$ $\alpha_1 f_1 + \alpha_2 f_2$ is convex for all $\alpha_1, \alpha_2 \geq 0$
  - the result extends to infinite sums and integrals
  - e.g., if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, and $w(y) \geq 0$ for each $y \in \mathcal{A}$, then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y) dy$$

    is convex in x (provided the integral exists)

Composition with an Affine Mapping

- f is convex $\implies$ $f(Ax + b)$ is convex
- if f is concave, so is $f(Ax + b)$
- read § 3.2.2 of Boyd's book

Pointwise Maximum and Supremum

- $f_1, f_2, \ldots, f_m$ are convex $\implies$ $\max\{f_1(x), f_2(x), \ldots, f_m(x)\}$ is convex
- if for each $y \in \mathcal{A}$ $f(x, y)$ is convex in x, then the function g, defined as

$$g(x) = \sup_{y \in \mathcal{A}} f(x.y)$$

  is convex inn x
- read § 3.2.3 of Boyd's book

Minimization

- if $f(x, y)$ is convex in (x, y), and $\mathcal{C}$ is a convex nonempty set, then the function g, defined as

$$g(x) = \inf_{y \in \mathcal{C}} f(x.y)$$

  is convex inn x

- examples:
  - suppose $f(x, y) = x^{\mathsf{T}} A x + 2x^{\mathsf{T}} B y + y^{\mathsf{T}} C y$, is convex in (x, y), [note $A, B \in S^n$] and $C \succeq 0$ what is $g(x)$? what can you say about $A - BC^{-1}B^{\mathsf{T}}$
  - is $\texttt{dis}(x, S) = \inf_{y \in \mathcal{S}} ||y - x||$, if $\mathcal{S}$ is convex?
- read § 3.2.5 of Boyd's book

Perspective

- the Perspective of a function $f : \mathbb{R}^n \to R$ is the function $g : \mathbb{R}^{n+1} \to \mathbb{R}$, defined by

$$g(x, t) = tf(x/t)$$

- if f is convex then g is convex
- examples: check what if $f(x) = x^{\mathsf{T}}x$, $f(x) = -\log(x)$
- read § 3.2.6 of Boyd's book

Composition with scalar functions

- read § 3.2.4, pages 83-84 of Boyd's book

How to Show a Given Function is Convex

- apply the definition
  - we pick any arbitrary two distinct points, say x, y from $\texttt{dom}\ f$
  - create an arbitrary convex combination z of those two points, i.e.,
$$z = \theta x + (1 - \theta)y, \quad \theta \in \mathbb{R}$$
  - show that the inequality (1) is satisfied, i.e.,
$$f(z) \leq \theta f(x) + (1 - \theta) f(y)$$

How to Show a Given Function is Convex

- verify the Hessian $\nabla^2 f(x)$ is positive semidefinite
  - $\nabla^2 f(x)$ is positive semidefinite, denoted $\nabla^2 f(x) \succeq 0$ for all nonzero $x \in \mathbf{dom}\ f \iff f$ is convex [1]
- restrict the function to an arbitrary line and verify the convexity of the resulting function of one variable
- argue the convexity by using operations that preserve convexity
- for details, refer to § 3.2, § 3.3 of ref-3

[1] In the case of $\mathbb{R}$ valued functions on $\mathbb{R}$, $\nabla^2 f(x) = d^2 f(x)/dx^2$.