YANGON TECHNOLOGICAL UNIVERSITY

Final Examination - September 2019

Introduction to Mathematical Optimization

(Two Hours)

Instructions to Candidates:

1. This is an open-book examination.
2. The notation is consistent with that used in lectures.
3. This paper has 3 questions.
4. An additional 10 minutes reading time is given.
5. Answer *all* questions.
6. This paper contains 3 pages.

Question 1 (Convexity of Sets and Functions) ..... [32pts]

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(1) Which of the following sets are convex? Justify your answer.

(a) $\mathcal{S} = \{x \mid a^{\mathrm{T}}x \leq b, c^{\mathrm{T}}x \geq d\}$, where $a, c \in \mathbb{R}^n$ and $b, d \in \mathbb{R}$. [8pts]

(b) $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$, where $\mathcal{S}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 + 1 \geq 0, x_2 \geq 0\}$ and $\mathcal{S}_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 + 1 \leq 0, x_2 \leq 0\}$. [8pts]

(c) $\mathcal{S} = \{(x, t) \in \mathbb{R}^{n+1} \mid ||x||_2 \leq t\}$.
Hint: You may use the *triangular* inequality $||a + b||_2 \leq ||a||_2 + ||b||_2$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$. [8pts]

(2) Consider the function $f(x) = x^{\mathrm{T}}Px + q^{\mathrm{T}}x + r$, where P is a real $n \times n$ symmetric *positive definite* matrix [i.e., $P \succ 0$], $q \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Show that f is convex by restricting it to a line. [8pts]

QUESTION 2 (KKT OPTIMALITY CONDITIONS) ..... [30pts]

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}} P^{-1} x + q^{\mathrm{T}} x + r \\ \text{subject to} & l \preceq x \preceq u, \end{array} \tag{1}$$

where the decision variable is $x \in \mathbb{R}^2$,

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad r = 1, \quad l = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and the vector inequality $\preceq$ above is to be intended elementwise.

(1) Sketch the feasible set. [8pts]

(2) Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (1). [5pts]

(3) Derive the KKT optimality conditions for the problem above. [8pts]

(4) Let $\tilde{\lambda} \in \mathbb{R}^2$ and $\tilde{\mu} \in \mathbb{R}^2$ denote the Lagrange multipliers associated with the constraint $l - x \preceq 0$ and $x - u \preceq 0$, respectively. It is given that $\tilde{\lambda} = (0, 0), \tilde{\mu} = (0, 5/4)$ are *dual optima*l. Derive the point $\tilde{x}$ that satisfy the KKT conditions [5pts]

(5) Does $x^\star = \tilde{x}$ hold ? Justify your answer. [4pts]

QUESTION 3 (LAGRANGE DUALITY) .....[38pts]

Consider the following convex optimization problem

$$\begin{array}{ll} \text{minimize} & a^{\mathrm{T}} x \\ \text{subject to} & \frac{1}{2}\, x^{\mathrm{T}} P x \leq 0, \end{array} \tag{2}$$

with variable $x \in \mathbb{R}^2$. The real 2×2 matrix P is symmetric and positive definite. Moreover, is given that $a \in \mathbb{R}^2$.

(1) Let $\lambda \in \mathbb{R}$ denote the Lagrange multiplier associated with the constraint. What is the Lagrangian $L(x, \lambda)$ associated with the problem (2) ? [4pts]

(2) Argue why the Lagrangian dual function g is unbounded below if $\lambda \leq 0$, i.e., $g(\lambda) = -\infty$ for $\lambda \leq 0$. [6pts]

(3) Let us denote by $x(\lambda)$ the minimizer of $L(x, \lambda)$ for $\lambda > 0$. In other words, $x(\lambda) = \underset{x \in \mathbf{R}^n}{\operatorname{argmin}} L(x, \lambda),\ \lambda > 0$. Show that [6pts]

$$x(\lambda) = -\frac{1}{\lambda} P^{-1} a.$$

Hint: Compute the gradient of L with respect to x.

(4) By using part (2) and part (3) above, show that the Lagrangian dual function g is given by [8pts]

$$g(\lambda) = \begin{cases} -\dfrac{1}{2\lambda}\left(10\lambda^2 + a^{\mathrm{T}} P^{-1} a\right) & \lambda > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(5) Derive a dual problem associated with problem (2). [4pts]

(6) If the matrix P and the vector a are given by

$$P = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{10} \\ \sqrt{10} \end{bmatrix},$$

show that $\lambda^\star = \sqrt{2}/3$, where $\lambda^\star$ is the dual optimal solution of the Lagrange dual problem of part (5). Determine the optimal solution $x^\star$ and the optimal value $p^\star$ of problem (2). [10pts]

Hint: Compute the gradient of g with respect to λ.

End of the Question Paper