Engineering Decision Theory, Quiz 9

University of Moratuwa, December 2019

Problem 1

1. Which of the following sets are convex? Justify your answer.

   (a) $\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\}$, where $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$.
   (b) $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = (-\infty, 5]$ and $\mathcal{S}_2 = \{6\}$.

2. Consider the set $\mathcal{X} \subseteq \mathbb{R}^2$ defined by the following inequalities:

$$[(x_1 \geq x_2 - 1) \text{ and } x_2 \geq 0] \text{ or } [(x_1 \leq x_2 - 1) \text{ and } x_2 \leq 0]\,.$$

   (a) Draw the set $\mathcal{X}$.
   (b) Is it convex? Justify your answer.
   (c) Show that it can be described as a single quadratic inequality of the form $q(x) = x^{\mathrm{T}}Ax + 2b^{\mathrm{T}}x + c \leq 0$, for a matrix $A = A^{\mathrm{T}} \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$ and $c \in \mathbb{R}$ which you will determine.
   (d) What is the convex hull of $\mathcal{X}$?

Problem 2

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined

$$f(x, y) = \max_{z>0} \Big(z(x + y) - y \ln(z) \Big). \tag{1}$$

   (a) For $x \in \mathbb{R}$ and $y \in \mathbb{R}_{++}$, show that $f(x, y) = y \exp(x/y)$.
   (b) By using the definition (1) above, argue why f is convex.

2. Consider the function $f(x) = ||x - b||_2^2$, where $b \in \mathbb{R}^m$. Show that f is convex by restricting it to a line. Recall that $||\cdot||_2$ denotes that ℓ_2-norm and $||a||_2^2 = a^{\mathrm{T}}a$.

Problem 3

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}}P^{-1}x + q^{\mathrm{T}}x + r \\ \text{subject to} & l \leq x \leq u, \end{array} \tag{2}$$

where the decision variable is $x \in \mathbb{R}^2$,

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, \quad q = \begin{bmatrix} -10 \\ -2/3 \end{bmatrix}, \quad r = 26, \quad l = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

1. Derive the KKT optimality conditions for the problem above.
2. Check whether the primal variable
$$\tilde{x} = (2, 1)$$
and the dual variables
$$\tilde{\lambda} = (0, 0), \quad \tilde{\mu} = (6, 4/9)$$
satisfy the KKT optimality conditions for the problem (2) above, where $\tilde{\lambda}$ is the Lagrange multipliers associated with $l - x \leq 0$ and $\tilde{\mu}$ is the Lagrange multipliers associated with $x - u \leq 0$.
3. Can $\tilde{x}$ be optimal for the primal problem (2)? Justify your answer.
4. Sketch the feasible set.
5. Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (2). Compare the $x^\star$ with $\tilde{x}$.

Problem 4

Consider the following problem:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \alpha_i \ln x_i \\ \text{subject to} & 1^{\mathrm{T}} x = c \\ & x \succeq 0, \end{array} \tag{3}$$

where the decision variable is $x \in \mathbb{R}^n$. Determine in closed-form a minimizer, and show that the optimal objective value of this problem is

$$p^\star = \alpha \ln(c/\alpha) + \sum_{i=1}^n \alpha_i \ln \alpha_i,$$

where $\alpha = \sum_{i=1}^n \alpha_i$. Steps of the derivation are to be clearly indicated.

Problem 5

A set of n jammers transmit with powers $p_1, \ldots, p_n$, where p_i is the power of i-th jammer. Of course, the transmit powers are nonnegative. In addition, the transmit power of each jammer is limited by the maximum transmit power $p_{\max}$. The jammers produce interference power at m receivers, given by

$$d_i = \sum_{j=1}^n G_{ij} p_j, \quad i = 1, \ldots, m,$$

where G_{ij} is the nonnegative channel gain from jammer j to receiver i. The channel gains G_{ij} for all i, j are known to the jammers. You may assume that the receiver i has a rate R_i (in bits/s) given by

$$R_i(d_i) = \log\left(1 + \frac{\beta_i}{\sigma^2 + d_i}\right), \quad i = 1, \ldots, m,$$

where β_i is the received signal power and σ^2 is the noise power at receiver i.

(1) Show that R_i is convex in d_i. (8pts)
Hint: Show first that $\log(1 + 1/x)$ *is convex for* $x > 0$.

(2) Explain how to choose $p = (p_1, \ldots, p_n)$ to minimize the sum rate R given by

$$R(d_1, \ldots, d_m) = \sum_{i=1}^{m} R_i(d_i).$$

using *convex* optimization. You may assume that the parameters β_i and σ^2 of R_i for all i are known by the jammers and jammers can perform the optimization jointly. The formulation of the optimization problem, together with the objective function and the associated constraint functions are to be explicitly mentioned.
Hint: The decision variables of the above problem is p *and* $(d_1, \ldots, d_m)$.