Engineering Decision Theory, Quiz 8

University of Moratuwa, December 2019

Problem 1

Let

$$c = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{d1} \end{bmatrix}, \; A = \begin{bmatrix} 3 & 1 \\ 5 & 4 \\ 1 & 1 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Now consider the problem

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}} x \\ \text{subject to} & A^{\mathrm{T}} x \preceq b \; , \end{array} \tag{1}$$

where the variable is $x \in \mathbb{R}^3$. Formulate (1) equivalently as a semidefinite program (SDP). Here you have to identify how to formulate $A^{\mathrm{T}} x \leq b$ in a form of

$$x_1 F_1 + x - 2F_2 + x_3 F_3 + G \preceq 0,$$

with symmetric F_1, F_2, F_3, and G.

Problem 2

Express the Liner Program (LP)

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}} x \\ \text{subject to} & A^{\mathrm{T}} x \preceq b \; , \end{array} \tag{2}$$

with variable $x \in \mathbb{R}^n$ as a SDP.

Problem 3

Consider the following Quadratic Program (QP):

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}} H x + p^{\mathrm{T}} x \\ \text{subject to} & Ax \succeq b \; , \end{array} \tag{3}$$

with variable $x \in \mathbb{R}^n$. It is given that $H \in S_+^n$.

1. Derive the epigraph form of the above problem by using $t \in \mathbb{R}$ as the auxiliary variable. You have to clearly mention the decision variables. (Hint: See problem (4.11) of [Boyd-2004].)

2. Show that the positive semidefinite matrix H can be written as $H = \hat{H}^{\mathrm{T}} \hat{H}$ for some $\hat{H} \in \mathbb{R}^{n \times n}$ (Hint: You may use the eigenvalue decomposition of H.)

3. Use part 2 above to show that

$$x^{\mathrm{T}}Hx + p^{\mathrm{T}}x \leq t \iff (t - p^{\mathrm{T}}x) - (\hat{H}x)^{\mathrm{T}}(\hat{H}x) \geq 0. \tag{4}$$

4. Express the right hand side condition of (4), i.e.,

$$(t - p^{\mathrm{T}}x) - (\hat{H}x)^{\mathrm{T}}(\hat{H}x) \geq 0, \tag{5}$$

as

$$\begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix} \succeq 0,$$

for some A, B, and C, which are to be defined appropriately. (Hint: See Appendix A and use that $B = \hat{H}x$.)

5. The part 4 above simply shows that (5) can be expressed as a Linear Matrix Inequality (LMI). By using the results of part 1, 2, 4, and that $Ax \succeq b$ can also be expressed as a LMI, derive the SDP formulation of QP (3). (Hint: See Appendix B and stack the variable t [see part 1] and x to form $\bar{x} = [t \;\; x]^{\mathrm{T}}$.)

Problem 4

Consider the following SDP:

$$\begin{array}{ll} \text{minimize} & \mathtt{tr}\,(CX) \\ \text{subject to} & \mathtt{tr}\,(A_iX) = b_i, \quad i = 1, \ldots, m, \\ & X \succeq 0, \end{array} \tag{6}$$

where $X \in S^n$ is the decision variable, $C, A_i \in S^n$, and $b_i \in \mathbb{R}$ are problem data.

1. Derive the Lagrangian.
2. Derive the Lagrangian dual function.
3. Derive the Dual problem of (6) with explicit definitions of related dual constraints.
4. Derive and state the KKT conditions for the SDP (6).

Problem 5

Write Matlab codes for computing the boundary of S^2_+ by using *two* methods:

1. Formulate an appropriate problem [say (P)] and solve it within the CVX environment.
2. Solve the problem (P) above by using KKT conditions.

Appendices

A Appendix (Schur Complement)

Given a matrix $X \in S^n$ partitioned as

$$X = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix},$$

with $A \in S^k$ and $\mathtt{det}\ A \neq 0$, the matrix $S = C - B^{\mathrm{T}} A^{-1} B$ is called the *Schur Complement* of A in X [§ A.5.5][Boyd-2004]. The following equivalences holds:

$$(X \succ 0) \Longleftrightarrow (A \succ \text{ and } S \succ 0),$$

$$(A \succ 0) \Longrightarrow (X \succeq 0 \Longleftrightarrow S \succeq 0).$$

B Appendix (Many LMIs as a Single LMI)

Two LMIs denoted

$$F_1(\bar{x}) \preceq 0 \text{ and } F_2(\bar{x}) \preceq 0$$

can be equivalently expressed as

$$\mathtt{diag} \begin{bmatrix} F_1(\bar{x}) & 0 \\ 0 & F_1(\bar{x}) \end{bmatrix} \preceq 0.$$

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