Engineering Decision Theory, Quiz 7

University of Moratuwa, December 2019

Problem 1

1. Which of the following sets are convex? Justify your answer.

   (a) $\mathcal{S} = \{x \mid a^{\mathrm{T}}x \leq b, c^{\mathrm{T}}x \geq d\}$, where $a, c \in \mathbb{R}^n$ and $b, d \in \mathbb{R}$.
   (b) $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$, where $\mathcal{S}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 + 1 \geq 0, x_2 \geq 0\}$ and $\mathcal{S}_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 + 1 \leq 0, x_2 \leq 0\}$.
   (c) $\mathcal{S} = \{(x, t) \in \mathbb{R}^{n+1} \mid ||x||_2 \leq t\}$.
   Hint: You may use the *triangular* inequality $||a+b||_2 \leq ||a||_2 + ||b||_2$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$.

2. Consider the function $f(x) = x^{\mathrm{T}}Px + q^{\mathrm{T}}x + r$, where P is a real $n \times n$ symmetric *positive definite* matrix [i.e., $P \succ 0$], $q \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Show that f is convex by restricting it to a line.

Problem 2

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}}P^{-1}x + q^{\mathrm{T}}x + r \\ \text{subject to} & l \preceq x \preceq u, \end{array} \tag{1}$$

where the decision variable is $x \in \mathbb{R}^2$,

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad r = 1, \quad l = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and the vector inequality $\preceq$ above is to be intended elementwise.

- Sketch the feasible set.
- Determine geometrically the optimal solution $x^\star$ and the optimal value $p^\star$ of the problem (1).
- Derive the KKT optimality conditions for the problem above.
- Let $\tilde{\lambda} \in \mathbb{R}^2$ and $\tilde{\mu} \in \mathbb{R}^2$ denote the Lagrange multipliers associated with the constraint $l - x \preceq 0$ and $x - u \preceq 0$, respectively. It is given that $\tilde{\lambda} = (0, 0), \tilde{\mu} = (0, 5/4)$ are *dual optimal*. Derive the point $\tilde{x}$ that satisfy the KKT conditions
- Does $x^\star = \tilde{x}$ hold ? Justify your answer.

Problem 3

Consider the following convex optimization problem

$$\begin{array}{ll} \text{minimize} & a^{\mathrm{T}}x \\ \text{subject to} & \frac{1}{2}\, x^{\mathrm{T}}Px \leq 5, \end{array} \tag{2}$$

with variable $x \in \mathbb{R}^2$. The real 2×2 matrix P is symmetric and positive definite. Moreover, is given that $a \in \mathbb{R}^2$.

1. Let $\lambda \in \mathbb{R}$ denote the Lagrange multiplier associated with the constraint. What is the Lagrangian $L(x, \lambda)$ associated with the problem (2) ? [4pts]

2. Argue why the Lagrangian dual function g is unbounded below if $\lambda \leq 0$, i.e., $g(\lambda) = -\infty$ for $\lambda \leq 0$. [6pts]

3. Let us denote by $x(\lambda)$ the minimizer of $L(x, \lambda)$ for $\lambda > 0$. In other words, $x(\lambda) = \underset{x \in \mathbf{R}^n}{\operatorname{argmin}}\, L(x, \lambda),\ \lambda > 0$. Show that [6pts]

$$x(\lambda) = -\frac{1}{\lambda} P^{-1}a.$$

   Hint: Compute the gradient of L with respect to x.

4. By using part (2) and part (3) above, show that the Lagrangian dual function g is given by [8pts]

$$g(\lambda) = \begin{cases} -\dfrac{1}{2\lambda}\left(10\lambda^2 + a^{\mathrm{T}}P^{-1}a\right) & \lambda > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(5) Derive a dual problem associated with problem (2). [4pts]

(6) If the matrix P and the vector a are given by

$$P = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{10} \\ \sqrt{10} \end{bmatrix},$$

show that $\lambda^\star = \sqrt{2}/3$, where $\lambda^\star$ is the dual optimal solution of the Lagrange dual problem of part (5). Determine the optimal solution $x^\star$ and the optimal value $p^\star$ of problem (2). [10pts]

Hint: Compute the gradient of g with respect to λ.