Engineering Decision Theory, Quiz 6

University of Moratuwa, December 2019

Problem 1

1. Which of the following sets are convex? Justify your answer.

   (a) $\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\}$, where $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$.
   (b) $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = (-\infty, 5]$ and $\mathcal{S}_2 = \{6\}$.
   (c) $\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2/4 + x_2^2/16 \leq 1\}$.
   Hint: You may use the *Cauchy-Schwarz* inequality $a^{\mathrm{T}}b \leq ||a||_2||b||_2$, where $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$, i.e., $a_1b_1 + a_2b_2 \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$.

2. Consider the function $f(x) = ||x - b||_2^2$, where $b \in \mathbb{R}^m$. Show that f is convex by restricting it to a line. Recall that $||\cdot||_2$ denotes that ℓ_2-norm and $||a||_2^2 = a^{\mathrm{T}}a$.

Problem 2

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{array} \tag{1}$$

1. Make a sketch of the feasible set.

2. Is problem (1) convex? Justify your answer.

3. Determine the optimal solution of problem (1) by using the sketch in part (a) above.

4. Determine the optimal value of problem (1).

Problem 3

Let $f_0(x_1, x_2) = x_1^2 + x_2^2$ and $f_1(x_1, x_2) = -(x_1 + x_2) + 1$.

1. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_1(x_1, x_2) \leq 0. \end{array} \tag{2}$$

   (a) Verify whether problem (2) is convex or not.
   (b) By using a sketch, find the optimal solution $x^\star = (x_1^\star, x_2^\star)$ and the optimal value $p^\star$ of problem (2).

2. Let us now solve optimization problem (2) above by using the *Lagrangian machinery*.

   (a) Let $\lambda \geq 0$, $x = (x_1, x_2)$, and define the function $L : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ as

   $$L(x, \lambda) = f_0(x) + \lambda f_1(x), \tag{3}$$

   which is called the *Lagrangian* associated with problem (2). Compute the gradient of the Lagrangian *with respect to* $x = (x_1, x_2) \in \mathbb{R}^2$, i.e.,

   $$\nabla_x L(x, \lambda) = \begin{bmatrix} \dfrac{\partial L(x, \lambda)}{\partial x_1} \\ \dfrac{\partial L(x, \lambda)}{\partial x_2} \end{bmatrix}.$$

   (b) Show that the $x = (\lambda/2, \lambda/2)$ minimizes the Lagrangian (3).
   Hint: The Lagrangian is convex with respect to $x = (x_1, x_2)$.

   (c) Let us denote by $x(\lambda)$ the solution of part (b) above. Substitute $x(\lambda)$ in the Lagrangian to obtain the function g defined by

   $$g(\lambda) = L(x(\lambda), \lambda).$$

   Determine $g(\lambda)$ and sketch it for the range $\lambda \geq 0$.

   (d) The function g is called the *dual function*. Let $\lambda^\star$ denote the value of λ that *maximizes* g. The $\lambda^\star$ is called the *dual optimal solution*. By using the sketch of part (c) above, show that $\lambda^\star = 1$.

   (e) Compute $x(\lambda^\star)$ and $g(\lambda^\star)$. Show that $x(\lambda^\star)$ is equal to the optimal solution $x^\star$ of problem (2). Moreover show that $g(\lambda^\star)$ is equal to the optimal value $p^\star$ of problem (2) which is known as the *zero duality* for convex problems.