Engineering Decision Theory, Quiz 5

University of Moratuwa, December 2019

Problem 1

Consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{16}(x_1-2)^2+\frac{1}{4}(x_2-1)^2 \\ \text{subject to} & x_1+x_2-1=0, \end{array} \tag{1}$$

where the decision variable is $x=(x_1,x_2)\in\mathbb{R}^2$. Graphically solve (1) to find the optimal solution $x^\star\in\mathbb{R}^2$ and the optimal value $p^\star$ of the problem.

Problem 2

Let us now solve optimization problem (1) above by eliminating the linear equality constraints. This is a standard method done in general to eliminate linear equality constraints in an optimization problem. You may refer to p. 132 of [Boyd-2004] for more details. The key idea is to first, *parametrize* the set of solutions of the underlying linear equality constraints to yield an equivalent problem to the original problem, and then to solve the equivalent problem. Particularize to problem (1), we first need to parametrize the solution set of $x_1+x_2=1$.

1. Let $x_2=\alpha$. Express x_1 in terms of α.
2. By using α, parametrize the solution set of $x_1+x_2=1$. In other words, what is x_1 in terms of α and x_2 in terms of α.
3. Express the objective function of problem (1) in terms of α.
4. Show that problem (1) now can be equivalently expressed as an *unconstrained optimization problem*, where the decision variable is α.
5. Determine the optimal solution $\alpha^\star$ of the unconstrained optimization problem in part 4 above.
6. By using $\alpha^\star$ in part 5 above, determine $x_1^\star$ and $x_2^\star$, the optimal solution for problem (1).

Problem 3

Let us now solve optimization problem (1) above by using the *Lagrangian machinery*.

1. Let $\nu \in \mathbb{R}$, $x = (x_1, x_2)$, and define the function $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as

$$L(x, \nu) = \left(\frac{1}{16}(x_1 - 2)^2 + \frac{1}{4}(x_2 - 1)^2 \right) + \nu(x_1 + x_2 - 1), \tag{2}$$

   which is the *Lagrangian* [1] associated with problem (1). Compute the gradient of the Lagrangian *with respect to* $x = (x_1, x_2) \in \mathbb{R}^2$, i.e.,

$$\nabla_x L(x, \nu) = \begin{bmatrix} \dfrac{\partial L(x, \nu)}{\partial x_1} \\ \dfrac{\partial L(x, \nu)}{\partial x_2} \end{bmatrix}.$$

   Note that the Lagrangian L of problem (1) is *convex with respect to* x, because it is quadratic in x. By setting the gradient $\nabla_x L(x, \nu)$ equal to zero, compute the x that minimizes the Lagrangian.

2. Let $x(\nu) = (x_1(\nu), x_2(\nu))$ denote the solution of part 1 above. Substitute $x_1(\nu)$ and $x_2(\nu)$ in the equality constraint $x_1 + x_2 - 1 = 0$ of problem (1) to determine the parameter ν. In other words, compute ν such that $x_1(\nu) + x_2(\nu) - 1 = 0$.

3. By substituting the ν computed in part 2 above, determine $x(\nu)$. Compare your result with the solution $x^\star$ of problem (1) that you computed in Problem 1.

The Lagrangian machinery is an extremely important area in mathematical optimization theory. The machinery is applied in general to any mathematical optimization problem. You may refer to § 5 of [Boyd-2004] for more details.

[1] The Lagrangian in the case of general constrained optimization problems is discussed in § 5.1.1, [Boyd-2004].