Engineering Decision Theory, Quiz 10

University of Moratuwa, December 2019

Problem 1

1. Let $\mathcal{S}$ be a subset of $\mathbb{R}^2$ defined as $\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2/4 + x_2^2/16 \leq 1\}$. Show that the $\mathcal{S}$ is convex by using the definition of convexity. [10pts]
   Hint: You may use the *Cauchy-Schwarz* inequality $a^{\mathrm{T}}b \leq ||a||_2||b||_2$.

2. Consider the set of points $\mathcal{V}$ that are closer (in Euclidean norm) to x_0 than x_i, $i = 1, \ldots, m$. That is, for any $x \in \mathcal{V}$, we have

$$||x - x_0||_2 \leq ||x - x_i||_2, \quad i = 1, \ldots, m.$$

   Show that $\mathcal{V}$ is a polyhedron by expressing it in the form $\mathcal{V} = \{x \mid Ax \preceq b\}$, for a matrix A and a vector b which you will determine. Note that the vector inequality $\preceq$ is to be intended elementwise.

3. Show that $f(X) = \mathtt{tr}(X^{-1})$ is convex on $\mathtt{dom}\ f = \mathbf{S}^n_{++}$ by restricting the function to a line.
   Hint: trace of the product of matrices is invariant under cyclical permutations, i.e., $\mathtt{tr}(AB) = \mathtt{tr}(BA)$, where the matrices A and B are of appropriate sizes.

Problem 2

Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & ||Ax - b||_2^2 \\ \text{subject to} & Gx = h, \end{array} \tag{1}$$

where the decision variable is $x \in \mathbb{R}^n$. The problem parameters A is a real $m \times n$ matrix with $\mathtt{rank}\ A = n$, G is a real $p \times n$ matrix with $\mathtt{rank}\ G = p$, b is a real m-vector, and h is a real p-vector.

1. Determine the Lagrangian associated with the problem (1).

2. Give KKT conditions for the problem.

3. By using KKT conditions, derive analytical expressions for the primal solution $x^\star \in \mathbb{R}^n$ and the dual solution $\nu^\star \in \mathbb{R}^p$, in terms of problem parameters.

Problem 3

Consider the following optimization problem

$$\begin{array}{ll} \text{minimize} & a^{\mathrm{T}}x \\ \text{subject to} & \frac{1}{2}\, x^{\mathrm{T}}Px \leq r, \end{array} \tag{2}$$

with variable $x \in \mathbb{R}^n$. The problem parameters $P \in S^n_{++}$, $a \in \mathbb{R}^n$, and $r \in \mathbb{R}$.

1. Let $\lambda \in \mathbb{R}$ denote the Lagrange multiplier associated with the constraint. What is the Lagrangian $L(x, \lambda)$ associated with the problem (2) ?

2. Show that the Lagrangian dual function g is given by

$$g(\lambda) = \begin{cases} -\dfrac{1}{2\lambda}\left(2r\lambda^2 + a^{\mathrm{T}}P^{-1}a\right) & \lambda > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

3. What is the associated dual problem.

4. Consider a particular case of the problem (2), with $n = 2$ and P, a, and r are given by

$$P = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}. \quad a = \begin{bmatrix} \sqrt{10} \\ \sqrt{10} \end{bmatrix}, \quad r = 5.$$

   Show that $\lambda^\star = \sqrt{2}/3$, where $\lambda^\star$ is the dual solution of the dual problem of part (5). Determine the primal solution $x^\star$ and the optimal value $p^\star$ of the problem (2).

Problem 4

Consider a transmitter at a base station (BS) who transmits signals to N receivers (users). The power allocated by the transmitter to the jth receiver is p_j, $j = 1, \ldots, N$. Let g_j denote the path gain from the transmitter to the jth receiver. The signal power at the jth receiver is given by $g_j p_j$. Moreover, the interference power at the jth receiver is given by $\sum_{i=1, i\neq j}^{N} g_j p_i$. The signal to interference plus noise ratio, denoted SINR, at the jth receiver is given by

$$\mathtt{SINR}_j(p) = \frac{g_j p_j}{\sigma_j^2 + \sum_{i=1, i\neq j}^{N} g_j p_i},$$

where σ_j^2 is the (self-) noise power at the jth receiver and $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$. The transmitter wants to allocate its *total available power* P_{T} so that the minimum SINR, over all the receivers, is maximized. In particular, the transmitter's power allocation problem can be formulated as

$$\begin{array}{ll} \text{maximize} & \min\limits_{j\in\{1,\ldots,N\}} \mathtt{SINR}_j(p) \\ \text{subject to} & 1^{\mathrm{T}}p \leq P_{\mathrm{T}} \\ & p \succeq 0, \end{array} \tag{3}$$

with variable p. Formulate the problem (3) equivalently as a *geometric program* (GP). The associated *posynomials* and *monomials* of the equivalent problem formulation are to be clearly identified.
Hint: Epigraph problem form is useful.