Engineering Decision Theory
EN5830

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Lecture 7: Convex Sets, [§ 2, Boyd-2018]

Line and Line Segment

- used when defining an affine set and a convex set
- graphically illustrate a line and a line segment

Line

- suppose $x_1 \neq x_2$ are two points in $\mathbb{R}^n$
- define the set

$$\mathcal{S} = \{\theta x_1 + (1-\theta)x_2 \mid \theta \in \mathbb{R}\}$$

- exercise:
  - draw the set when $x_2 = (1, 1)$ and $x_1 = (1, 2)$
  - draw the set when $x_2 = (0, 0)$ and $x_1 = (1, 0)$
  - what corresponds to $\theta = 0$?
  - what corresponds to $\theta = 1$?

Line Segment

- suppose $x_1 \neq x_2$ are two points in $\mathbb{R}^n$
- define the set

$$\mathcal{S} = \{\theta x_1 + (1-\theta)x_2 \mid \theta \in \mathbb{R}, \theta \in [0,1]\}$$

- exercise:
  - draw the set when $x_1 = (1,1)$ and $x_2 = (1,2)$
  - draw the set when $x_1 = (0,0)$ and $x_2 = (1,0)$
  - what corresponds to $\theta = 0$?
  - what corresponds to $\theta = 1$?

Affine Set

- a set $\mathcal{S}$ is affine whenever the line through any distinct points in $\mathcal{S}$ lies in $\mathcal{S}$
- exercise: show that the following sets are affine
  - $\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}\}$, i.e., the range of A
  - $\mathcal{N}(A) = \{x \mid Ax = 0\}$, i.e., the nullspace of A
  - $\mathcal{S} = \{x \mid Ax = b\}$, i.e., the set of solutions of $Ax = b$
- any combination of 2 points x_1, x_2 in an affine set $\mathcal{S}$, of the form $\theta x_1 + (1 - \theta)x_2$, lies in $\mathcal{S}$

Affine Combination

- we refer to a point of the form

$$\theta_1 x_1 + \ldots + \theta_k x_k \quad \text{where} \quad \theta_1 + \ldots + \theta_k = 1$$

  as an affine combination of the points $x_1, \ldots, x_k$

- any affine combination of points in an affine set $\mathcal{S}$, lies in $\mathcal{S}$

Affine Hull

- the set of all affine combinations of points in some set $\mathcal{S} \in \mathbb{R}^n$ is called the affine hull of $\mathcal{S}$, and denoted $\mathtt{aff}\ \mathcal{S}$:

  $$\mathtt{aff}\ \mathcal{S} = \{\theta_1 x_1 + \ldots + \theta_k x_k \mid \theta_1 + \ldots + \theta_k = 1,\ x_i \in \mathcal{S},\ i = 1, \ldots, k\}$$

- graphically illustrate the idea..

- the affine hull is the smallest affine set that contains $\mathcal{S}$, i.e.,

  $$\mathcal{A} \text{ is any affine set with } \mathcal{S} \subseteq \mathcal{A} \implies \mathtt{aff}\ \mathcal{S} \subseteq \mathcal{A}$$

Convex Set

- a set $\mathcal{S}$ is convex whenever the line segment through any distinct points in $\mathcal{S}$ lies in $\mathcal{S}$
- exercise: show that the following sets are convex
  - $\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}\}$, i.e., the range of A
  - $\mathcal{N}(A) = \{x \mid Ax = 0\}$, i.e., the nullspace of A
  - $\mathcal{S} = \{x \mid Ax = b\}$, i.e., the set of solutions of $Ax = b$
- linear segment between any 2 points in a convex set $\mathcal{S}$, lies in $\mathcal{S}$

Convex Combination

- we refer to a point of the form

  $\theta_1 x_1 + \ldots + \theta_k x_k \quad$ where $\quad \theta_1 + \cdots + \theta_k = 1$ and $\theta_1, \ldots, \theta_k \geq 0$

  as a convex combination of the points $x_1, \ldots, x_k$

- any convex combination of points in an convex set $\mathcal{S}$, lies in $\mathcal{S}$

- note: a point in a line segment is a particular case of a convex combination

Convex Hull

- the set of all convex combinations of points in some set $\mathcal{S} \in \mathbb{R}^n$ is called the convex hull of $\mathcal{S}$, and denoted $\mathbf{conv}\ \mathcal{S}$:

$$\mathbf{conv}\ \mathcal{S} = \left\{ \sum_{i=1}^{k} x_i \theta_i \;\middle|\; \sum_{i=1}^{k} \theta_i = 1,\ \theta_i \geq 0,\ x_i \in \mathcal{S},\ i = 1, \ldots, k \right\}$$

- graphically illustrate the idea..

Convex Cone

- a set $\mathcal{C}$ is a cone whenever the ray that originates at the origin and passes through any point in $\mathcal{C}$ lies in $\mathcal{C}$, i.e.,

$$\text{for every } x \in \mathcal{C} \text{ and } \theta \geq 0,\ \theta x \in \mathcal{C}.$$

- examples:
  - $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$,
    i.e., the set of nonnegative real numbers
  - nonconvex examples?
- a set $\mathcal{C}$ is a convex cone if it is convex and a cone, i.e.,

$$\text{for every } x_1, x_2 \in \mathcal{C} \text{ and } \theta_1, \theta_2 \geq 0,\ \theta_1 x_1 + \theta_2 x_2 \in \mathcal{C}.$$

Conic Combination

- we refer to a point of the form

$$\theta_1 x_1 + \ldots + \theta_k x_k \quad \text{with} \quad \theta_1, \ldots, \theta_k \geq 0$$

  as a conic combination of the points $x_1, \ldots, x_k$

- any conic combination of points in an convex cone $\mathcal{C}$, lies in $\mathcal{C}$

Conic Hull

- the set of all conic combinations of points in some set $\mathcal{C}$ is called the conic hull of $\mathcal{C}$
- i.e., the set

$$\left\{\sum_{i=1}^{k} x_i\theta_i \;\middle|\; \theta_i \geq 0,\ x_i \in \mathcal{S},\ i = 1, \ldots, k\right\}$$

  is the conic hull of $\mathcal{S}$
- graphically illustrate the idea..

Some Common Convex Sets

- empty set: the set of the form:

$$\mathcal{S} = \{\ \} = \emptyset$$

- singleton: any single point:

$$\mathcal{S} = \{x_0\}$$

- whole space: i.e.,

$$\mathcal{S} = \mathbb{R}^n$$

- any line
- any line segment
- any affine set

Some Common Convex Sets

- ray: the set of the form:

$$\mathcal{S} = \{x_0 + \theta v \mid \theta \geq 0\},\ x_0 \in \mathbb{R}^n,\ v \neq 0$$

- hyperplane: the set of the form:

$$\mathcal{S} = \{x \mid a^T x = b\},\ a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R}$$

  - note that the hyperplane define above is the level set of the function $f(x) = a^T x$ at level b

Some Common Convex Sets

- other interpretations of the hyperplane defined above: it is
  - the set of solutions of $a^T x = b$
  - the set of points with a constant inner product to a given vector a
  - a hyperplane with normal vector a and an offset $b/||a|| \in \mathbb{R}$ from the origin
  - the set of the form

$$\mathcal{S} = \{x \mid a^T(x - x_0) = 0\},$$

where $a \in \mathbb{R}^n,\ a \neq 0$, and $x_0 \in \mathbb{R}^n$ is a particular solution of linear system $a^T x = b$

Some Common Convex Sets

- halfspace: the set of the form:

$$\mathcal{S} = \{x \mid a^T x \leq b\},\ a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R}$$

  - note: a hyperplane divides $\mathbb{R}^n$ into two hapfspaces
  - note that the halfspace define above is the sub level set of the function $f(x) = a^T x$ at level b
  - it can also be represented by

$$\begin{aligned}\mathcal{S} &= \{x \mid a^T x \leq a^T x_0\},\ a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R} \\ &= \{x \mid a^T (x - x_0) \leq 0\}\end{aligned}$$

  - graphically illustrate the idea

Some Common Convex Sets

- norm ball: the ball with centre $x_c \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_+$ is set of the form:

$$\begin{aligned}\mathcal{B}(x_c, r) &= \{x \mid ||x - x_c|| \leq r\} \\ &= \{x \mid ||x - x_c||^2 \leq r^2\} \\ &= \{x \mid (x - x_c)^T (x - x_c) \leq r^2\} \\ &= \{x_c + ru \mid ||u|| \leq 1\}\end{aligned}$$

- graphically illustrate the idea

Some Common Convex Sets

- norm cone: is set of the form:

$$\mathcal{C} = \{(x, t) \mid ||x|| \leq t\} \subseteq \mathbb{R}^{n+1}$$

- graphically illustrate the idea

Some Common Convex Sets

- ellipsoid: the set of the form:

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\},$$

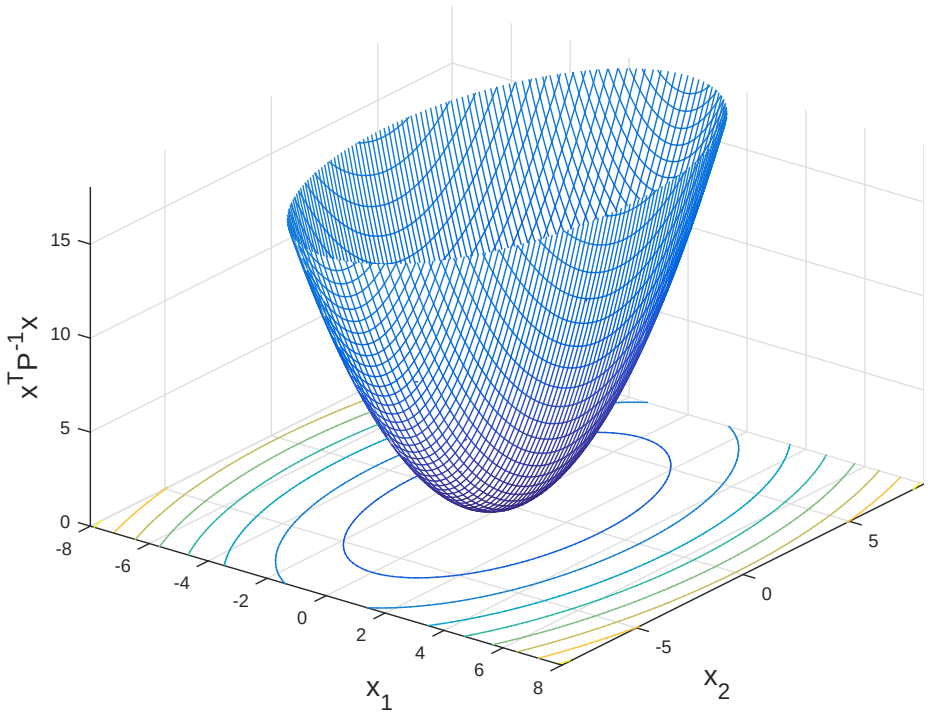
where P is a symmetric positive definite matrix [1], i.e.,

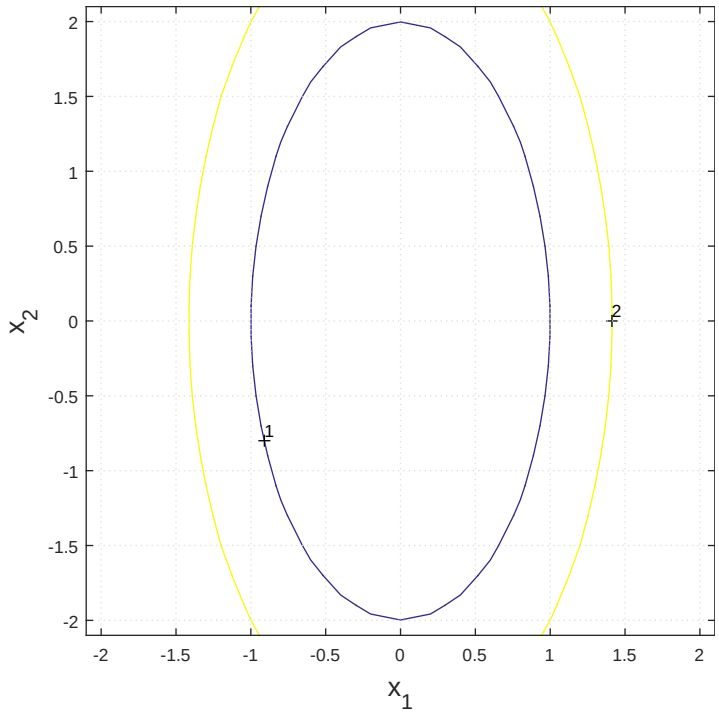
$$P \in \mathbb{S}^{n \times n} \quad \text{and} \quad \text{for all nonzero } x \in \mathbb{R}^n,\ x^T P x > 0$$

- graphically illustrate the idea starting with high school knowledge of the equation of ellipse

[1] We say a P is symmetric positive semidefinite whenever $P \in \mathbb{S}^{n \times n}$ and for all $x \in R^n,\ x^T P x \geq 0$

Some Common Convex Sets

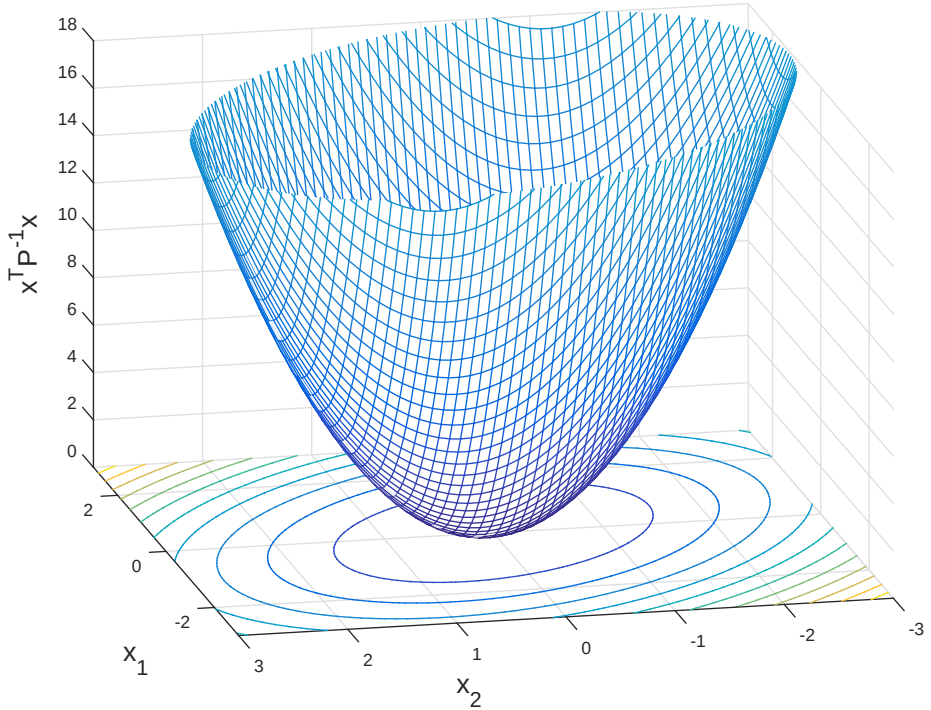


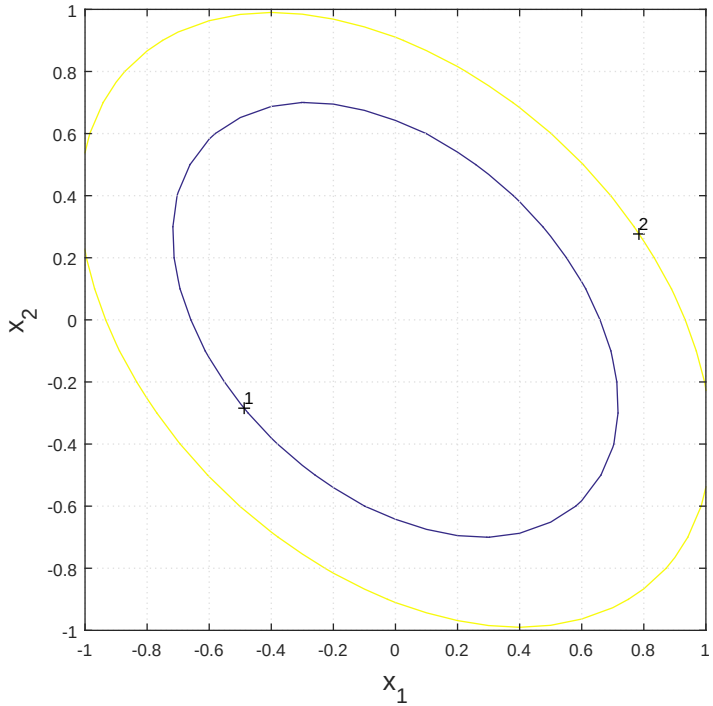


- $\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$, with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad x_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Some Common Convex Sets

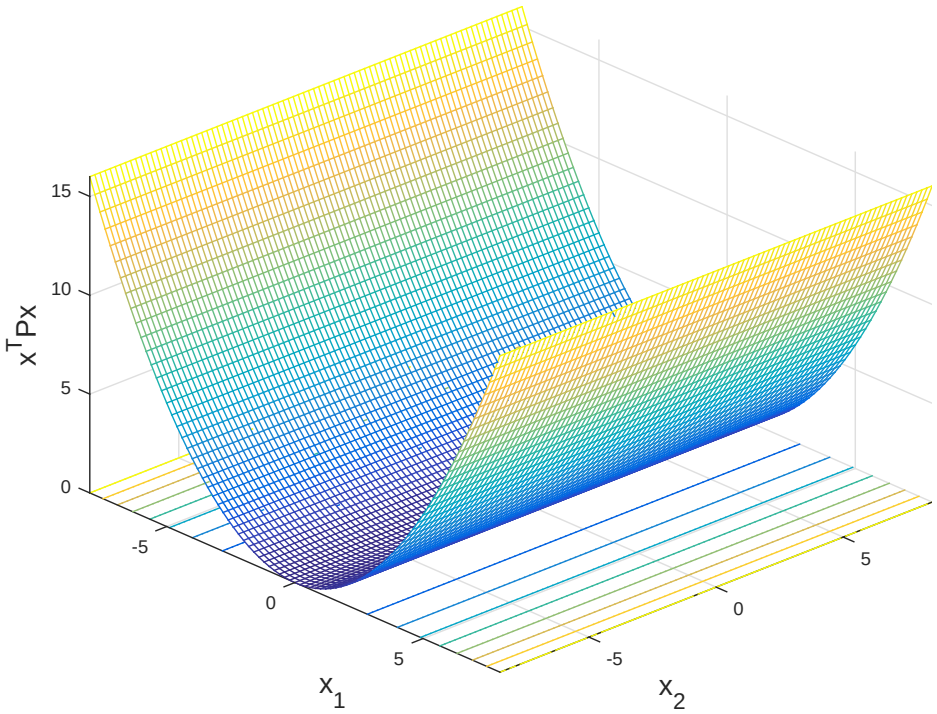


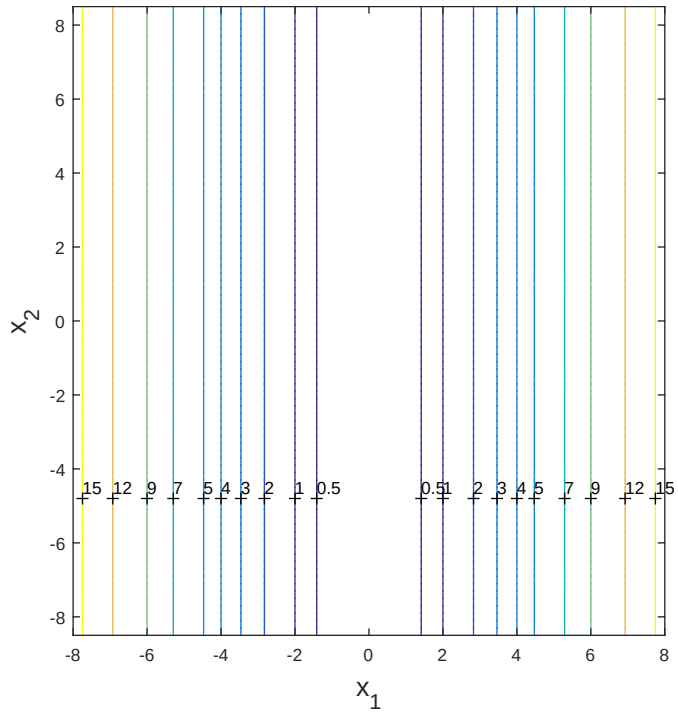


- $\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$, with

$$P = \begin{bmatrix} 0.5166 & -0.1978 \\ -0.1978 & 0.4908 \end{bmatrix}, \qquad x_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Some Common Convex Sets

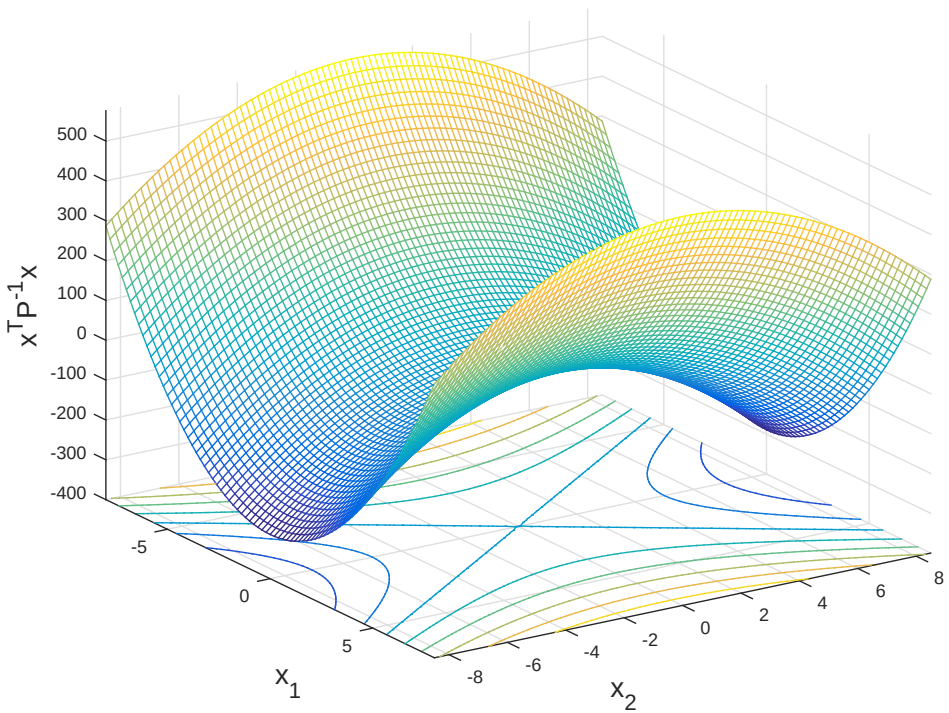


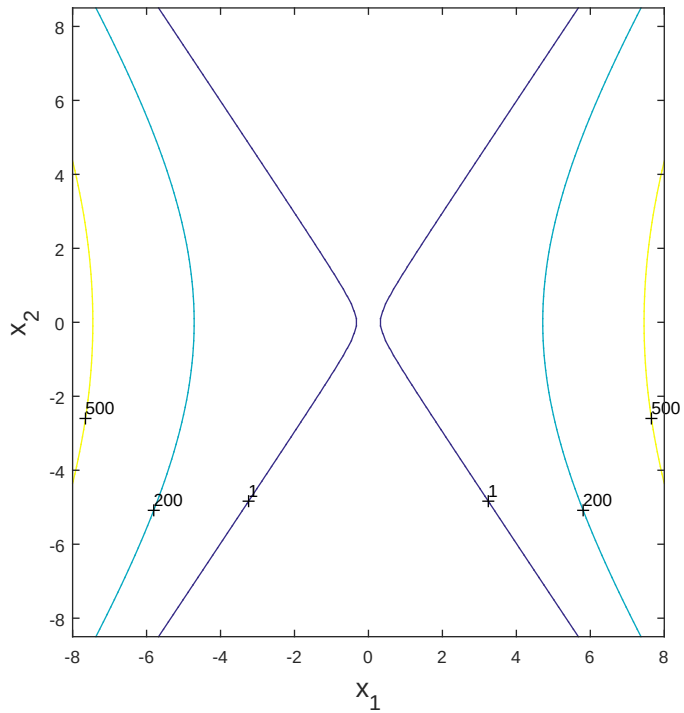


- $\mathcal{E} = \{x \mid (x - x_c)^T P (x - x_c) \leq 1\}$, with

$$P = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \end{bmatrix}, \qquad x_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Some Common Convex Sets





- $\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$, with

$$P = \begin{bmatrix} 9 & 0 \\ 0 & -4 \end{bmatrix}, \qquad x_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- note the nonconvecity of the set $\mathcal{E}$ whenever P is not positive indefinite

Some Common Convex Sets

- polyhedron: is the solution set of a finite number of linear equalities and inequalities:

$$\mathcal{P} = \{x \mid a_j^T x \leq b_j,\ j = 1, \ldots, m,\ c_j^T x = d_j,\ j = 1, \ldots, p\},$$

- graphically illustrate the idea

Some Common Convex Sets

- other interpretations of the polyhedron defined above:
  - it is he intersection of a finite number of halfspaces and hyperplanes
  - affine sets (e.g., subspaces, hyperplanes, lines), rays, line segments, and halfspaces are all polyhedra
  - a hyperplane with normal vector a and an offset $b/||a|| \in \mathbb{R}$ from the origin
  - the set of the form

$$\mathcal{P} = \{x \,|Ax \preceq b,\ Cx = d\}, \quad A \in \mathbb{R}^{m\times n}, \quad C \in \mathbb{R}^{p\times n}$$

    where the symbol $\preceq$ denotes the vector inequality or componentwise inequality in $\mathbb{R}^m$

Some Common Convex Sets

- let S^n denote the set of symmetric $n \times n$ matrices
- let S^n_+ denote the symmetric positive semidefinite matrices, i.e.,

$$S^n_+ = \{X \in S^n \mid X \succeq 0\}$$

- let S^n_{++} denote the symmetric positive definite matrices, i.e.,

$$S^n_{++} = \{X \in S^n \mid X \succ 0\}$$

- positive semidefinite cone: S^n_+ is a convex cone, called positive semidefinite cone
- what does S^n_{++} stand for?

Some Common Convex Sets

- there are many others
- for details you may refer to § 2.2 of [Boyd-2018]

How to Show a Given Set is Convex

- apply the definition
  - we pick arbitrary two distinct points, say x_1, x_2 from the set $\mathcal{S}$
  - create a convex combination y of those two points with an an arbitrary $\theta \in [0, 1]$, i.e.,

$$y = \theta x_1 + (1 - \theta) x_2, \quad \theta \in [0, 1]$$

  - show that y is in $\mathcal{S}$

How to Show a Given Set is Convex

- argue the convexity by using operations that preserve convexity
  - intersection
  - affine function
  - perspective function
  - linear-fractional functions
- for details, refer to § 2.3 of [Boyd-2018]

Intersection

- if $\mathcal{S}_1$ and $\mathcal{S}_2$ are convex, then $\mathcal{S}_1 \bigcap \mathcal{S}_2$ is convex
- if $\mathcal{S}_\alpha$ is convex for every $\alpha \in \mathcal{A}$, then $\bigcap_{\alpha \in \mathcal{A}} \mathcal{S}_\alpha$ is convex
- show that S_+^n is convex

Affine functions

- recall: a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine if it is a sum of a linear function and a constant, i.e.,

$$f(x) = Ax + b, \quad A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m.$$

- if $\mathcal{S} \subseteq \mathbb{R}^n$ is convex, then the image of $\mathcal{S}$ under f,

$$f(\mathcal{S}) = \{ f(x) \mid x \in \mathcal{S} \}$$

  is convex

- if $\mathcal{B} \subseteq \mathbb{R}^m$ is convex, then the inverse image of $\mathcal{B}$ under f,

$$f^{-1}(\mathcal{B}) = \{ x \mid f(x) \in \mathcal{B} \}$$

  is convex

Affine functions

- examples: show that
  - $\mathcal{E} = \{x \mid (x - x_c)^{\mathsf{T}} P^{-1}(x - x_c) \leq 1\}$ is convex, where $P \in S^n_{++}$
  - hint: think of the image of norm ball [with ℓ_2 norm] under $f(x) = P^{1/2}x + x_c$

Affine functions

- examples: show that
  - $\mathcal{S} = \{x \in \mathbb{R}^n \mid x^\mathsf{T} P x \leq (c^\mathsf{T} x)^2, c^\mathsf{T} x \geq 0\}$ is convex, where $P \in S_+^n$
  - hint: think of the inverse image of the second-order cone under $f(x) = (P^{1/2}x, c^\mathsf{T} x)$

Perspective function

- perspective function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$:

$$P(x,t) = \frac{x}{t}, \quad \mathbf{dom}\ P = \{(x,t) \mid t > 0\}$$

- images and inverse images of convex sets under P are convex

Linear-Fractional functions

- linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \mathbf{dom}\ f = \{x \mid c^T x + d > 0\}$$

- images and inverse images of convex sets under f are convex