Digital Communications
EN5601

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Lecture 9: Digital Modulation Schemes (Part II)

References

- the discussion is essentially based on [§ 6, Gallager-2008] and [§ 5, Benedetto-2008]

Modulation Schemes: Trade-offs in Choice

- Shannon's theories places bounds on the values of the above parameters that can be achieved by any modulation scheme

Channel Capacity

- the result is extremely general and is not restricted to Gaussian channels
- let us consider the the capacity theorem in the particular case of transmission over AWGN channel

Capacity Theorem

- there exists a constant, C_N, called Gaussian channel capacity, given by

$$C_N = \frac{1}{2}\log_2\left(1 + 2\,\frac{E_N}{N_0}\right) \qquad \text{[bits/dimension]},$$

  - N : dimension of the signal space
  - E_N: available energy per dimensions

Capacity Theorem

- moreover, C_N has the following properties:
  - $R_N > C_N$, equi-probable signals $M = 2^{NR_N}$ is large [1] $\implies$ $P[\mathcal{E}] \to 1$ for any set of transmitter signals
  - $R_N < C_N$, $M = 2^{NR_N}$ is sufficiently large [3] $\implies$ there exist sets of M transmitter signals such that $P[\mathcal{E}]$ with optimum receivers is arbitrarily small
- R_N transmitter input rate in bits per dimension

[1] M is made large by increasing N.

Capacity in Bits Per Second

- suppose D is the available dimensions per second
- what is the maximum value of D in a channel band-limited to W Hz ?

Capacity in Bits Per Second

- suppose D is the available dimensions per second
- what is the maximum value of D in a channel band-limited to W Hz ? $2W$

Capacity in Bits Per Second

- by multiplying both sides of the capacity formula, we get

$$C = 2W \;\; C_N = W \;\; \log_2\left(1 + 2\,\frac{E_N}{N_0}\right) \qquad \text{[bits/sec]}$$

$$= W \;\; \log_2\left(1 + 2\,\frac{P/(2W)}{N_0}\right)$$

$$= W \;\; \log_2\left(1 + \frac{P}{N_0 W}\right) \qquad \text{[bits/sec]}$$

  - note $E_N = P/(2W)$
  - note $P/(N_0 W) = (E_b/N_0)(R/W)|_{R=C} = (E_b/N_0)(C/W)$

Capacity in Bits Per Second

- i.e.,

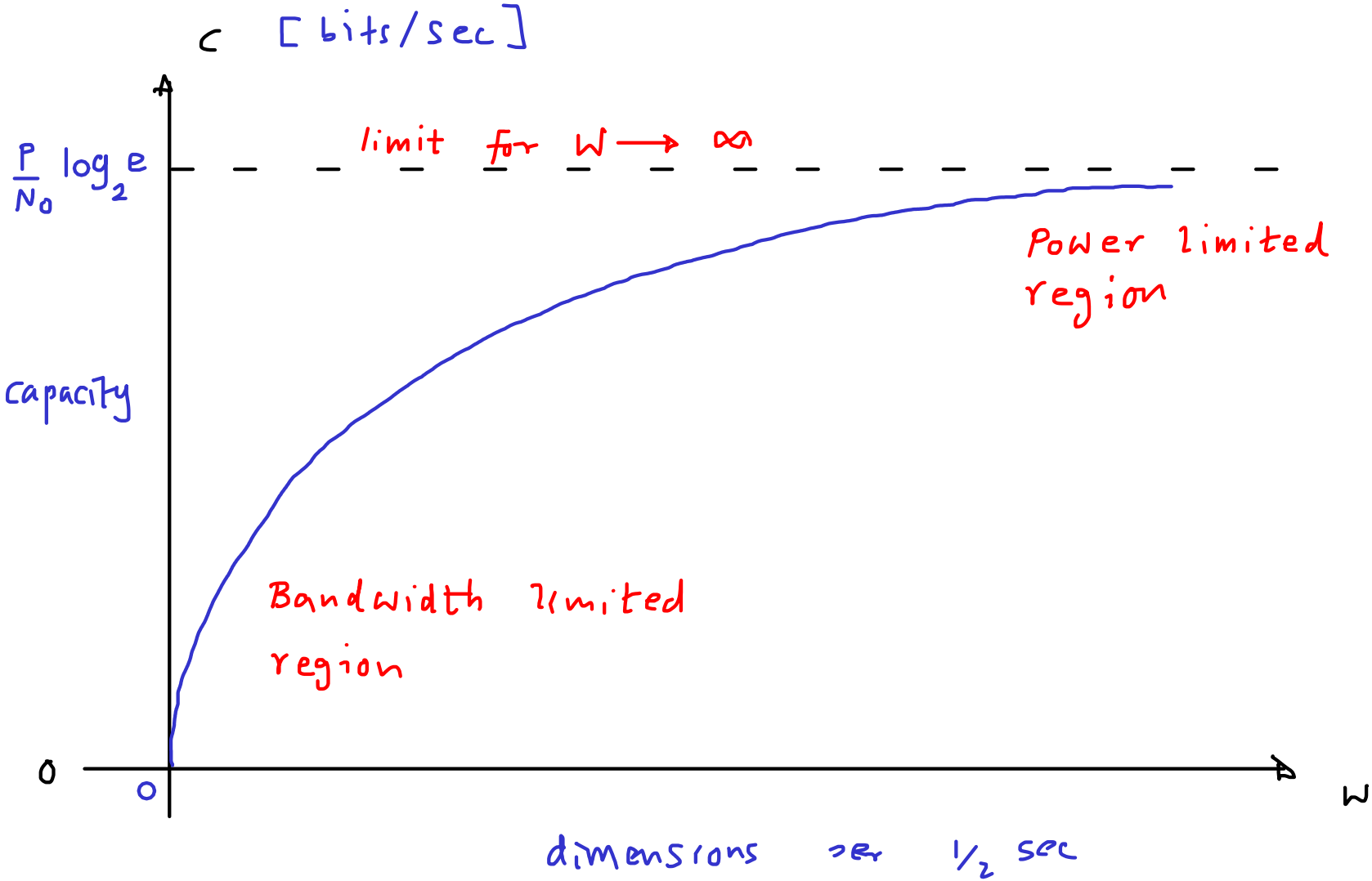
$$C = W \ \log_2\left(1 + \frac{P}{N_0 W}\right) \qquad \text{[bits/sec]} \qquad (1)$$

$$\frac{C}{W} = \log_2\left(1 + \frac{E_b}{N_0}\frac{C}{W}\right) \qquad \text{[bits/pair of dimensions]} \qquad (2)$$

Capacity in Bits Per Second

- draw the curve to represent all possible pairs of points $(C, W) \in \mathbb{R}^2$ characterized by (1)

Capacity in Bits Per Second



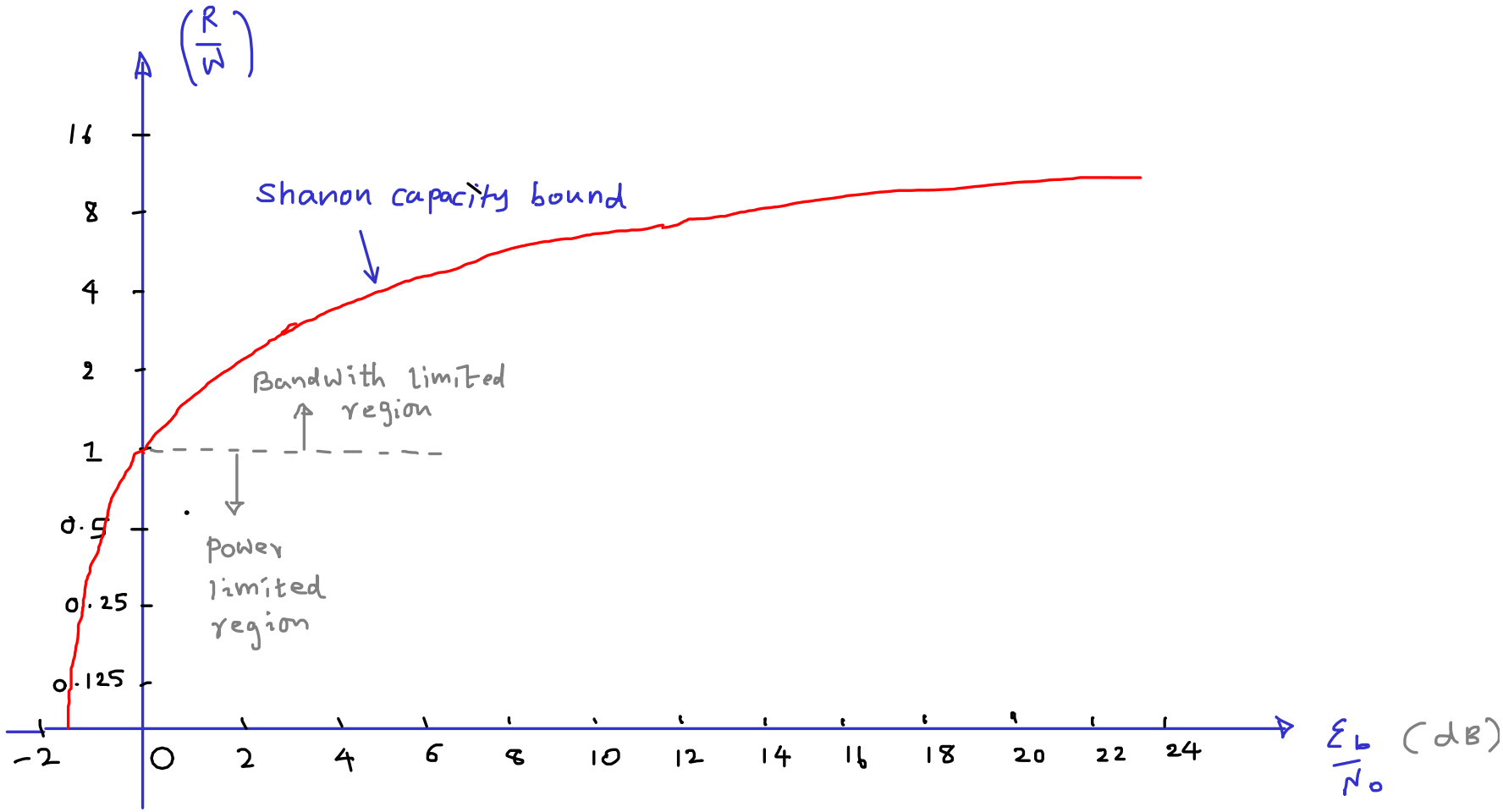
Capacity in Bits Per Second

- the equation (2) yields the following:

$$\frac{E_b}{N_0} = \frac{2^{C/W} - 1}{C/W} \tag{3}$$

- draw the curve to represent all possible pairs $(E_b/N_0, C/W) \in \mathbb{R}^2$ of points characterized by

Capacity in Bits Per Second



PAM

PAM, Baseband Representation

- the baseband representation x_{BB} is

$$x_{\mathrm{BB}}(t) = \sum_k u_k \; p(t - kT)$$

- here the constellation is $\mathcal{A} = \{a_1, a_2, \ldots, a_M\}$, where $M = 2^b$
- in particular,

$$\mathcal{A} = \left\{ \frac{-d(M-1)}{2}, \frac{-d(M-3)}{2}, \cdots, \frac{d(M-3)}{2}, \frac{d(M-1)}{2} \right\}$$

PAM, Error Probability

- the error probability of M-PAM is given by

$$P[\mathcal{E}] = 2\,\frac{M-1}{M} Q\left(\frac{d}{\sqrt{2N_0}}\right) \tag{4}$$

- recall that $E_s = E_b \log_2(M) = d^2(M^2-1)/12$
- therefore,

$$d = 2\sqrt{\frac{3\ \log_2(M)\ E_b}{M^2-1}}$$

- thus, $P[\mathcal{E}]$ can be written in terms of E_b/N_0 as follows:

$$P[\mathcal{E}] = 2\,\frac{M-1}{M} Q\left(\sqrt{\frac{6\ \log_2(M)}{(M^2-1)}\ \frac{E_b}{N_0}}\right) \tag{5}$$

PAM, Bandwidth Efficiency

- suppose the signalling interval is T
- the Shannon bandwidth, W is given by $W = 1/(2T)$ (i.e., dimensions per half a second)
- moreover, $R = (\log_2 M)/T$
- therefore, we have

$$\frac{R}{W} = 2 \log_2 M$$

- increasing M improves the BW efficiency, but decrease power efficiency

QAM

QAM, Baseband Representation

- the baseband representation of M-QAM is

$$x_{\mathrm{BB}}(t) = \sum_k u_k p(t - kT)$$

where $u_k \in \mathcal{A}$ and

$$\mathcal{A} = \left\{ a + ib \;\middle|\; a, b \in \tilde{\mathcal{A}} \right\} \tag{6}$$

where

$$\tilde{\mathcal{A}} = \left\{ \frac{-d(\sqrt{M}-1)}{2}, \frac{-d(\sqrt{M}-3)}{2}, \cdots, \frac{d(\sqrt{M}-3)}{2}, \frac{d(\sqrt{M}-1)}{2} \right\}$$

QAM, Baseband Representation

- when $p(t)$ is real, we have

$$\text{Re}\left\{x_{\text{BB}}(t)\right\} = \sum_k \text{Re}\{u_k\}p(t-kT)$$

$$\text{Im}\left\{x_{\text{BB}}(t)\right\} = \sum_k \text{Im}\{u_k\}p(t-kT)$$

- M-point square constellation $\rightarrow$ two independent $\sqrt{M}$-PAM

- recall that the E_s is given by

$$\begin{aligned} E_s &= 2 \times (\text{energy per } \sqrt{M}\text{-PAM signal}) \\ &= \frac{d^2[M-1]}{6} \end{aligned}$$

QAM, Error Probability

- based on the above properties, probability of correct decision $P[\mathcal{C}]$ in square M-QAM is square of the probability of correct decision for $\sqrt{M}$-PAM

- i.e.,

$$\begin{aligned} P[\mathcal{E}] &= 1 - P[\mathcal{C}] = 1 - \left(P_{\text{PAM}}[\mathcal{C}]\right)^2 \\ &= 1 - \left(1 - P_{\text{PAM}}(\mathcal{E})\right)^2 \\ &= 1 - \left(1 - 2\,\frac{\sqrt{M}-1}{\sqrt{M}} Q\left(\sqrt{\frac{3\ \log_2 M}{(M-1)}\ \frac{E_b}{N_0}}\right)\right)^2 \end{aligned} \tag{7}$$

- note that E_b remains same for both QAM and PAM [2]

[2] Here the vertical and the horizontal distances of signals in QAM is same as the distance between signals in PAM.

QAM, Error Probability

- some approximations: assume large M and large E_b/N_0
- from (7)

$$
\begin{aligned}
P[\mathcal{E}] &= 1 - (1 - P_{\mathrm{PAM}}(\mathcal{E}))^2 = 2P_{\mathrm{PAM}}(\mathcal{E}) - (P_{\mathrm{PAM}}(\mathcal{E}))^2 \\
&\leq 2P_{\mathrm{PAM}}(\mathcal{E}) \\
&= 2\,\frac{\sqrt{M}-1}{\sqrt{M}} Q\left(\sqrt{\frac{3\ \log_2 M}{(M-1)}\ \frac{E_b}{N_0}}\right) \\
&\leq 2\ Q\left(\sqrt{\frac{3\ \log_2 M}{(M-1)}\ \frac{E_b}{N_0}}\right)
\end{aligned}
$$

QAM, Bandwidth Efficiency

- suppose the signalling interval is T
- the Shannon bandwidth W is given by $W = 2/(2T) = 1/T$
- moreover, $R = (\log_2 M)/T$
- therefore, we have

$$\frac{R}{W} = \log_2 M$$

- increasing M improves the BW efficiency, but decrease power efficiency

Phase-Shift Keying (PSK)

PSK, Baseband Representation

- the baseband representation of M-PSK is

$$x_{\mathrm{BB}}(t) = \sum_k u_k p(t - kT)$$

where $u_k \in \mathcal{A}$ and

$$\mathcal{A} = \left\{ \sqrt{E} e^{i\phi} \;\middle|\; \phi \in \tilde{\mathcal{A}} \right\} \tag{8}$$

where

$$\tilde{\mathcal{A}} = \left\{ \frac{2\pi}{M}(n-1) \;\middle|\; n = 1, 2, \ldots, M \right\}$$

PSK, Baseband Representation

- when $p(t)$ is real, we have

$$\mathrm{Re}\left\{x_{\mathrm{BB}}(t)\right\} = \sum_k \mathrm{Re}\{u_k\}p(t-kT)$$

$$\mathrm{Im}\left\{x_{\mathrm{BB}}(t)\right\} = \sum_k \mathrm{Im}\{u_k\}p(t-kT)$$

- the energy per signal E_s is given by

$$E_s = E$$

PSK, Error Probability

- binary PSK (2-PSK), which is a binary modulation scheme
- $P[\mathcal{E}]$ can be deduced in an straightforward manner:

$$P[\mathcal{E}] = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$
$$= Q\left(\frac{2\sqrt{E}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- note that for 2-PSK, $E = E_b$

PSK, Error Probability

- 4-PSK can be deduced by considering 4-QAM modulation scheme

$$
\begin{aligned}
P[\mathcal{E}] &= 1 - (1-p)^2 \\
&= 2p - p^2 \\
&= 2\ Q\left(\frac{d}{\sqrt{2N_0}}\right) - \left[Q\left(\frac{d}{\sqrt{2N_0}}\right)\right]^2 \\
&= 2\ Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2
\end{aligned}
$$

- note that for 4-PSK, $d = \sqrt{2E} = 2\ \sqrt{E_b}$

PSK, Error Probability

- M-PSK: no closed-form expressions for $P[\mathcal{E}] \rightarrow$ bounds on $P[\mathcal{E}]$
- $P[\mathcal{E}]$ can be bounded [3]

$$Q\left(\sqrt{\frac{2E_b}{N_0}\log_2 M}\ \sin\left(\frac{\pi}{M}\right)\right) \leq P[\mathcal{E}] \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}\log_2 M}\ \sin\left(\frac{\pi}{M}\right)\right)$$

- note that $E_b = E/\log_2(M)$

[3] Details of the derivation is in § 4.3.1 of [Benedetto-2008]

PSK, Bandwidth Efficiency

- suppose the signalling interval is T
- the Shannon bandwidth W is given by $W = 2/(2T) = 1/T$
- moreover, $R = (\log_2 M)/T$
- therefore, we have

$$\frac{R}{W} = \log_2 M$$

- increasing M improves the BW efficiency, but decrease power efficiency

Orthogonal Frequency-Shift Keying (FSK)

FSK, Passband Representation

- the passband M-FSK signal is given by

$$x_{\mathrm{PB}}(t) = 2\ \mathrm{Re}\left\{\sum_k p(t-kT)e^{i2\pi f_d \xi_k (t-kT)} e^{i2\pi f_c t}\right\}$$

  where ξ_k takes the values $\pm 1, \pm 3, \ldots, \pm(M-3), \pm(M-1)$

- label the index k from 1 to M [4]
- thus, $\xi_k \in \{2n-1-M \mid n \in \{1,2,\ldots,M\}\}$
- for each n, associate $(2n-1-M)$ to a signal $s_n(t)$

[4] M is an even number.

FSK, Vector View

- the signal $s_n(t)$, $n = 1, 2, \ldots, M$ is defined as

$$\begin{aligned} s_n(t) &= \cos\left(2\pi(f_0 + f_d\xi)t\right) \\ &= \cos\left(2\pi[f_0 + f_d(2n - 1 - M)]t\right) \end{aligned}$$

  for $0 \leq t \leq T$, and 0 otherwise

- by choosing appropriately the frequency parameter f_d, the signals above can be made almost orthogonal

- show that $\langle s_n, s_m \rangle \approx 0$ for all $n \neq m$ when $f_d T = 1/4$

FSK, Vector View

- as a result, for each time index k, the underlying signal can be represented by the set of basis functions $\{\phi_n(t)\}_{n=1}^{M}$, where

$$\phi_n(t) = \frac{s_n(t)}{||s_n(t)||}$$

- i.e., the underlying signals (for each time index k) is spanned by $\{\phi_n(t)\}_{n=1}^{M}$

- in particular, $\xi = 2n - 1 - M$ is mapped to an M-vector s_n whose all but the nth element is nonzero, i.e.,

$$s_n = \sqrt{E}\, e_n,$$

where e_n is the standard nth unit vector and E is the energy of signal $s_n(t)$

FSK, Constellation

- sketch the constellation of FSK, for $M = 2$

FSK, Error Probability

- M-FSK: no closed-form expressions for $P[\mathcal{E}] \rightarrow$ bounds on $P[\mathcal{E}]$
- $P[\mathcal{E}]$ can be bounded [p. 239, Benedetto-2008]

$$P[\mathcal{E}] \leq (M-1)\, Q\left(\sqrt{\frac{E_b}{N_0}\log_2 M}\right)$$

- note that $E_b = E/\log_2(M)$

FSK, Bandwidth Efficiency

- suppose the signalling interval is T
- the Shannon bandwidth W is given by $W = M/(2T)$
- moreover, $R = (\log_2 M)/T$
- therefore, we have

$$\frac{R}{W} = 2\frac{\log_2 M}{M}$$

- unlike with PAM, PSK, and QAM, by increasing M the BW efficiency of FSK decreases
- on the other hand, its power efficiency increases

Digital Modulation Trade-Offs

Bandwidth Efficiency and E_b/N_0

- bandwidth efficiency and E_b/N_0 of modulation schemes

Bandwidth Efficiency and E_b/N_0

