Digital Communications
EN5601

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Lecture 8: Digital Modulation Schemes (Part I)

References

- the discussion is essentially based on [§ 6, Gallager-2008] and [§ 5, Benedetto-2008]

Key Performance Metrics

- bandwidth
- power
- error probability

Bandwidth

Several Definitions

- see p. 217 of [Benedetto-2008]
  - null-to-null bandwidth
  - fractional power-constrained bandwidth: frequency interval that contain $(1-\epsilon)$ of the signal power
  - bounded power-spectral-density bandwidth: everywhere outside the BW, the power spectral density doesnot exceed a certain threshold
- however it is convenient to have an abstract definition for BW

Abstract Definition

- $2WT$- theorem [1]
- the space of signals of approximate duration T and approximate bandwidth W has approximate dimension $2WT$
- i.e., any function within that class can be specified approximately by specifying about $2WT$ real numbers as coefficients in an orthogonal expansion
- let $\{\psi_i(t)\}$ be any set of orthogonal waveforms of duration T and bandwidth $W \longrightarrow$ the number of waveforms in $\{\psi_i(t)\}$ is upperbounded by $2WT$

[1] For more details, see pp. 80-82 of [Benedetto-2008], § 4.7.2 of [Gallager-2008], or, Appendix 5A of [Wozencraft-1965] .

Shannon Bandwidth

- thus, the dimension of signals with duration T and bandwidth W is approximately $N = 2WT$
- Shannon bandwidth is defined as

$$W = \frac{N}{2T}$$

where $2W$ is of units dimensions per second [dimensions/sec]

Power

Recall (Already Discussed)

- R is data rate [bits/sec]
- M symbols
- M symbols $\longrightarrow$ M signals
- signal rate is $R_s = R/\log_2(M)$ [signals/sec]
- signal duration (interval) is T [sec]
- i.e., $R = \log_2(M)/T$ [bits/sec]

Transmit Power

- in the case of detection (coming lectures), we deal with the transmission of a single signal
- so we will have signals subject to a constraint on the transmit energy, E_s
- but when a sequence of signals are considered, energy constraint is no longer meaningful
- it is meaningful and instructive to impose a bound on the average transmit power, P

Transmit Power

- for a signal $s(t)$ of duration T the average signal power is given by

$$P = \frac{1}{T}\int_0^T |s(t)|^2 dt = \frac{1}{T}\int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{E_s}{T} \qquad (1)$$

- a constraint on power $P \implies$ available transmit energy grows no more than linearly with time

- the average energy available per bit, denoted E_b is given by

$$E_b = \frac{P}{R} \qquad \text{[jouls/bit]} \qquad (2)$$

Signal to Noise Ratio (SNR)

- the ratio between the average signal power and the average noise power
- the avarage noise power = noise power density $\times$ bandwidth $= (N_0/2)(2W) = N_0 W$
- one may interpret $N_0/2$ as noise energy per dimension, i.e., N_0 is the noise energy per two dimensions [p. 253, Gallager-2008]

Signal to Noise Ratio (SNR)

- thus,

$$\text{SNR} = \frac{P}{N_0 W} = \frac{(E_b\ R)}{(N_0\ W)} = \left(\frac{E_b}{N_0}\right)\ \left(\frac{R}{W}\right) \tag{3}$$

- the term (R/W) of above is referred to as bandwidth efficiency

Bandwidth Efficiency

- i.e.,

$$\text{bandwidth efficiency} = \frac{R}{W} \qquad [(\text{bits/sec})/\text{Hz}]$$

- units of bandwidth efficiency : (bits/sec) per Hz
- one may also interpret (R/W) in terms of bits per two dimensions



- we also have

$$\frac{R}{W} = \frac{\log_2(M)/T}{N/(2T)} = 2\,\frac{\log_2(M)}{N}$$

Error Probability [2]

[2] See [§ 4.5, Wozencraft-2008]

Big Picture: AWGN Channel

- transmitter signals are represented by the constellation with M points
- those M points are in an N-dimensional space
- the relevant noise disturbance is represented by a random vector $V \in \mathbb{R}^N$ whose pdf is given by

$$p_V(v) = \frac{1}{(\pi N_0)^{N/2}} \, e^{-||v||^2/N_0}$$

Big Picture: AWGN Channel

- the optimum receiver divides the signal space into M disjoint regions $\{I_i\}_{i=1,\ldots,M}$
- any received point ρ is assigned to I_k if and only if

$$||\rho - s_k||^2 - N_0 \ln P[m_k] < ||\rho - s_i||^2 - N_0 \ln P[m_i] \qquad \forall\, i \neq k$$

- different constellations $\longrightarrow$ different probability error expressions

Rectangular Constellations

Binary Signals

- binary signals (equi-probable) who are d distance units apart

$$P[\mathcal{E}] = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

- what if the signals are antipodal?
- what if the signals are orthogonal

Rectangular Decision Regions

- optimum decision regions are always rectangular when the constellation is rectangular
- we may use rotation and translation appropriately if needed
- draw a figure and explain

Vertices of a Hypercube

- when $M = 2^N$ signals are located on the vertices of an N-dimensional hypercube
- draw a figure and explain
- technically, we have

$$s_k = (s_{k1}, \ldots, s_{kN}), \qquad k = 1, \ldots, 2^N,$$

where $s_{kj} \in \{-d/2, d/2\}$ for all j

Vertices of a Hypercube

- event of correct decision conditioned on that $s_1 = (-d/2, \ldots, -d/2)$ is transmitted?
- draw a figure and explain
- more specifically,

$$P\left[\, \mathcal{C} \mid s_1 \,\right] = P\left[\, \forall\ k,\ W_k \leq d/2 \,\right] = \prod_k P\left[\ \ W_k \leq d/2 \,\right]$$

$$= \prod_k \left(1 - P\left[\ \ W_k > d/2 \,\right]\right) = \prod_k \left(1 - Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)$$

$$= (1-p)^N$$

Vertices of a Hypercube

- by symmetry, $P[\mathcal{C}] = (1-p)^N$
- i.e., $P[\mathcal{E}] = 1-(1-p)^N$, where $p = Q\big(d/(\sqrt{(}2N_0))\big)$
- we also have energy per signal $E_s = Nd^2/4$, i.e.,

$$p = Q\left(\sqrt{\frac{2E_s}{NN_0}}\right)$$

Orthogonal and Related Constellations

Orthogonal and Related Constellations

- another class of equally likely signals for which the $P[\mathcal{E}]$ is easily computed
- e.g.,
  - orthogonal signals
  - simplex signal
  - biorthogonal signals
- details are deferred to latter lectures

Bottom Line

- bandwidth, power, and error probability is different for different modulation schemes
- trade-offs are there

Trade-offs in Modulation Schemes

Modulation Schemes: Trade-offs in Choice

- the parameters that we derived can be used to discuss different modulation schemes
- e.g.,
  - $P[\mathcal{E}]$
  - SNR
  - E_b/N_0
  - R/W
- e.g., one modulation scheme is good in one parameter, while bad in another

Modulation Schemes: Trade-offs in Choice

- Shannon's theories places bounds on the values of the above parameters that can be achieved by any modulation scheme

Channel Capacity

- the result is extremely general and is not restricted to Gaussian channels
- let us consider the the capacity theorem in the particular case of transmission over AWGN channel

Capacity Theorem

- there exists a constant, C_N, called Gaussian channel capacity, given by

$$C_N = \frac{1}{2}\log_2\left(1 + 2\,\frac{E_N}{N_0}\right) \qquad \text{[bits/dimension]},$$

  - N : dimension of the signal space
  - E_N: available energy per dimensions

Capacity Theorem

- moreover, C_N has the following properties:
  - $R_N > C_N$, equi-probable signals $M = 2^{NR_N}$ is large [3] $\Longrightarrow$ $P[\mathcal{E}] \to 1$ for any set of transmitter signals
  - $R_N < C_N$, $M = 2^{NR_N}$ is sufficiently large [3] $\Longrightarrow$ there exist sets of M transmitter signals such that $P[\mathcal{E}]$ with optimum receivers is arbitrarily small
- R_N transmitter input rate in bits per dimension

[3] M is made large by increasing N.

Capacity in Bits Per Second

- suppose D is the available dimensions per second
- what is the maximum value of D in a channel band-limited to W Hz ?

Capacity in Bits Per Second

- suppose D is the available dimensions per second
- what is the maximum value of D in a channel band-limited to W Hz ? $2W$

Capacity in Bits Per Second

- by multiplying both sides of the capacity formula, we get

$$C = 2W \;\; C_N = W \;\; \log_2\left(1 + 2\,\frac{E_N}{N_0}\right) \qquad \text{[bits/sec]}$$

$$= W \;\; \log_2\left(1 + 2\,\frac{P/(2W)}{N_0}\right)$$

$$= W \;\; \log_2\left(1 + \frac{P}{N_0 W}\right) \qquad \text{[bits/sec]}$$

  - note $E_N = P/(2W)$
  - note $P/(N_0 W) = (E_b/N_0)(R/W)|_{R=C} = (E_b/N_0)(C/W)$

Capacity in Bits Per Second

- i.e.,

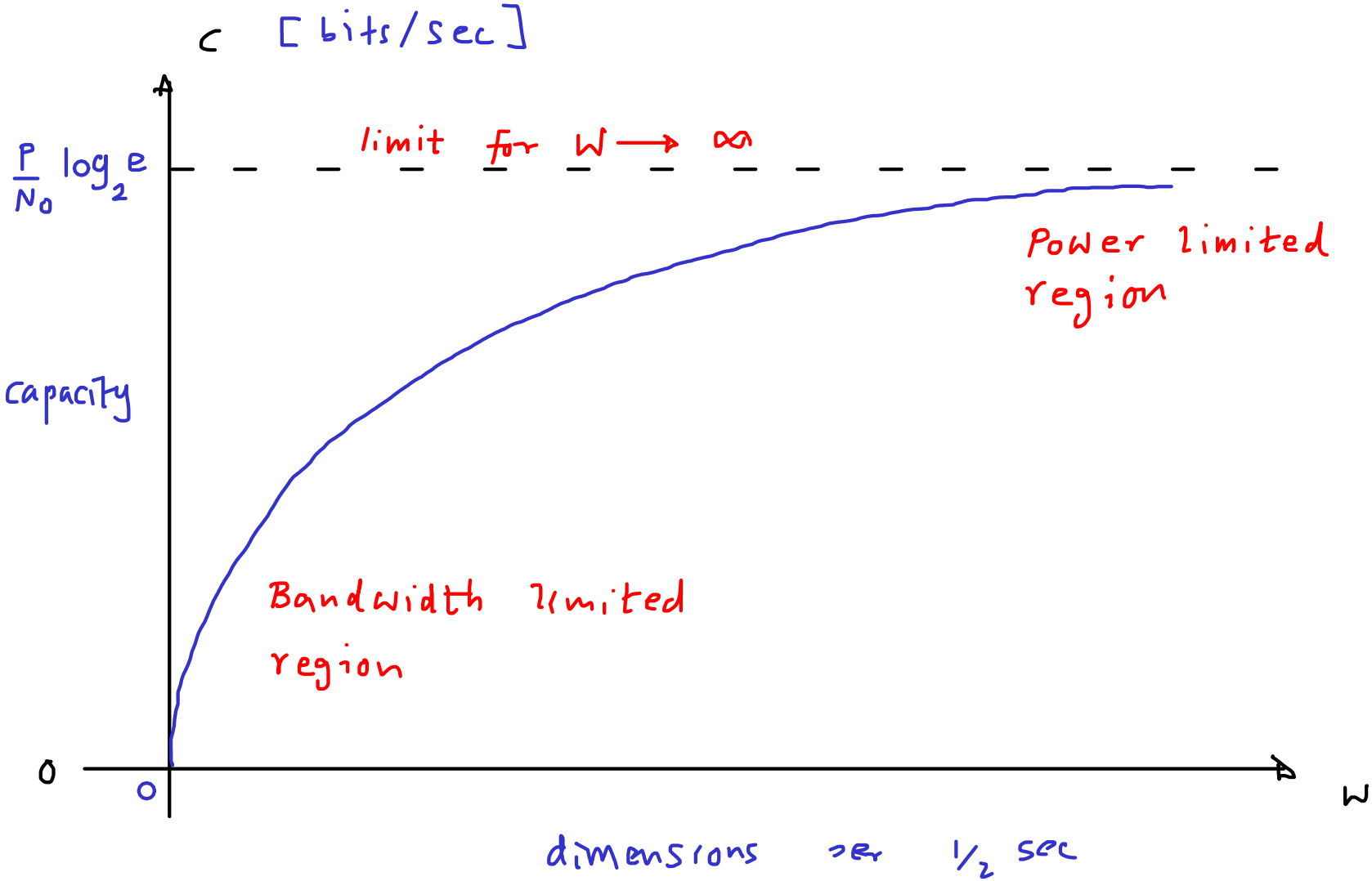
$$C = W \ \log_2\left(1 + \frac{P}{N_0 W}\right) \qquad \text{[bits/sec]} \qquad (4)$$

$$\frac{C}{W} = \log_2\left(1 + \frac{E_b}{N_0}\frac{C}{W}\right) \qquad \text{[bits/pair of dimensions]} \qquad (5)$$

Capacity in Bits Per Second

- draw the curve to represent all possible pairs of points $(C, W) \in \mathbb{R}^2$ characterized by (4)

Capacity in Bits Per Second



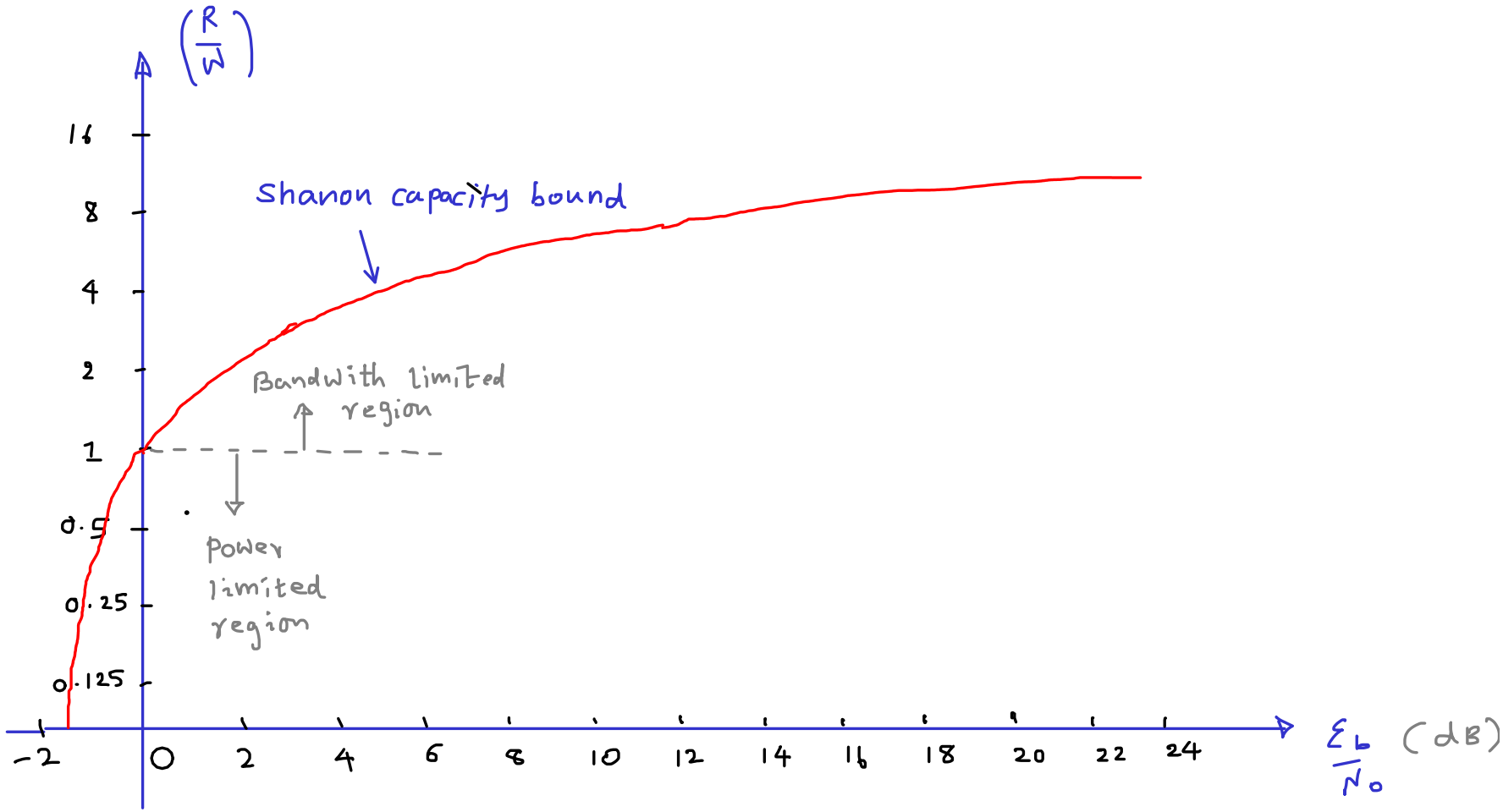
Capacity in Bits Per Second

- the equation (5) yields the following:

$$\frac{E_b}{N_0} = \frac{2^{C/W} - 1}{C/W} \tag{6}$$

- draw the curve to represent all possible pairs $(E_b/N_0, C/W) \in \mathbb{R}^2$ of points characterized by

Capacity in Bits Per Second



Capacity in Bits Per Second

- next lecture: the trade-offs of main modulation techniques and where those reside w.r.t Shannon's limits