Digital Communications
EN5601

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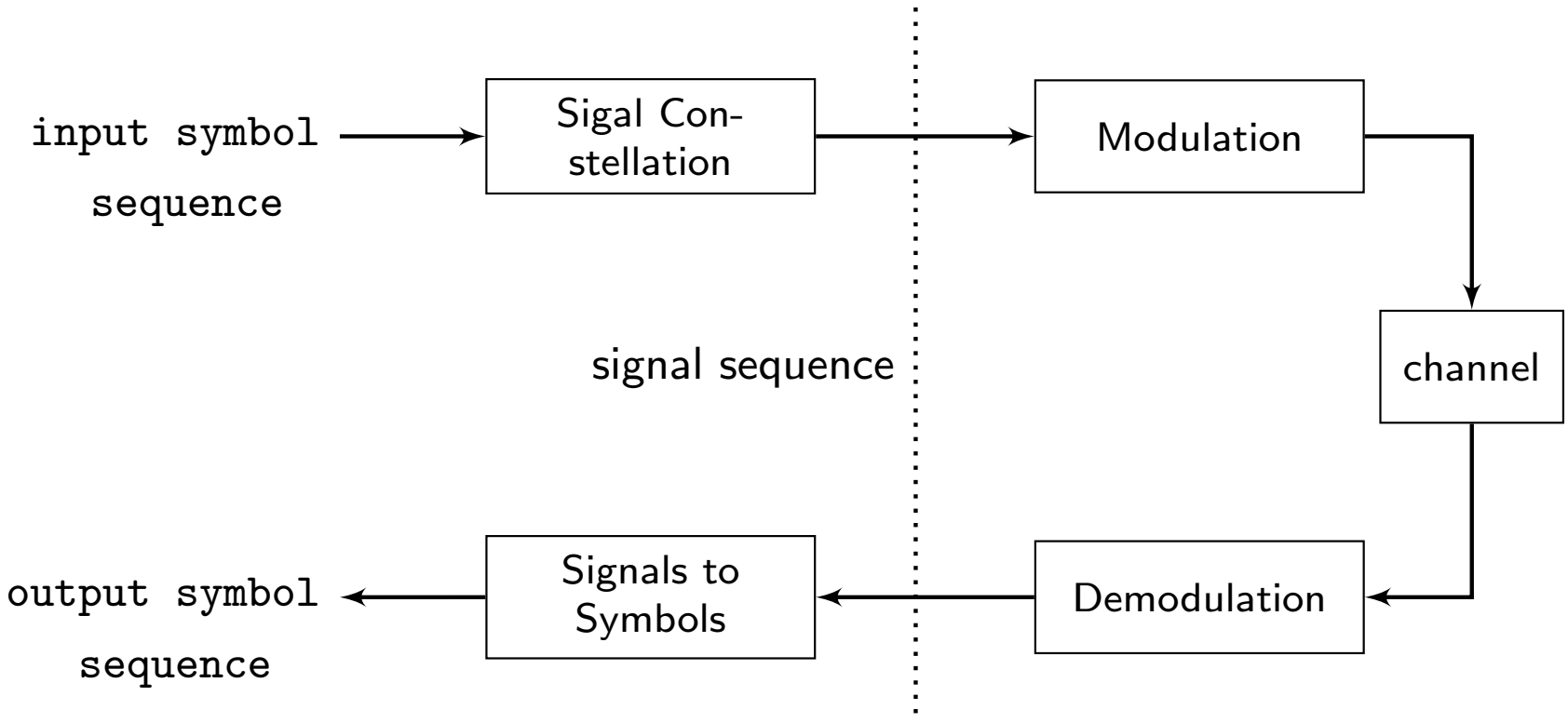
Lecture 6: Channels, Modulation, and Demodulation (part II)

References

- the discussion is essentially based on [§ 6, Gallager-2008] and [§ 7, Lapidoth-2017] [1]

[1] '*Amos Lapidoth, 'A Foundation in Digital Communication*', 2nd edition, Cambridge University Press, 2017

Channel Encoding / Decoding



Modulation Block

- modulation block usually has 2 layers: signals to baseband waveforms $\rightarrow$ baseband to passband
- converting baseband waveform into a passband waveform?
- appropriate for the physical medium, regulatory constraints, and avoiding other transmission bands

A Real Baseband Signal

What if the Baseband Signal is Real?

- e.g., let the baseband waveform be $u(t)$ the PAM waveform
- modulating $u(t)$ to some high-frequency passband around f_c
  - shift $u(t)$ up in frequency to yield $x(t) = u(t)e^{2\pi i f_c t}$
- result is real or complex ?

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  - shift $u(t)$ up in frequency to yield $x(t) = u(t)e^{2\pi i f_c t}$
- result is real or complex ? complex waveform
- but we need a real passband: only real waveforms can be transmitted
- make $x(t)$ conjugate-symmetric by adding further the term $\left(u(t)e^{2\pi i f_c t}\right)^* = u(t)e^{-2\pi i f_c t}$

What if the Baseband Signal is Real?

- the resulting passband waveform $x(t)$ is

$$\begin{aligned} x(t) &= u(t)e^{2\pi i f_c t} + u(t)e^{-2\pi i f_c t} \\ &= 2\ u(t)\cos(2\pi f_c t) \end{aligned}$$

- since $u(t)$ is real, these two bands are redundant and either could be reconstructed from the other

- thus, the above passband signal is quite wasteful: half of the band is redundant

Complex Baseband Signal

- can we do better?

Complex Baseband Signal

- can we do better? yes,

Complex Baseband Signal

- can we do better? yes, how?
- make the baseband signal not conjugate-symmetric
- i.e., make the baseband signal is complex

Passband Waveform

Passband Waveform

- signals encountered in wireless communications are typically real passband signals
- such signals can be represented using their complex baseband representation
- the discussion and the results holds for $\mathcal{L}_1$ passband signals
- most of the results will also hold for $\mathcal{L}_2$ passband signals [2]

[2] For a rigorous treatment on the topic, one may refer to § 7 of '*A Foundation in Digital Communication*', 2nd edition, Cambridge University Press, 2017, by Amos Lapidoth.

A Real Passband Signal

A Real Passband Signal

- let $x_{\mathrm{BB}}(t)$ be a baseband signal, whose FT $\hat{x}_{\mathrm{BB}}(f)$ is not necessarily hermitian-symmetric
- i.e., $x_{\mathrm{BB}}(t)$ is a complex baseband signal
- the FT of real passband signal $\hat{x}_{\mathrm{PB}}(t)$ of $x_{\mathrm{BB}}(t)$ is

$$\hat{x}_{\mathrm{PB}}(f) = \hat{x}_{\mathrm{BB}}(f - f_c) + \hat{x}^*_{\mathrm{BB}}(-f - f_c)$$

A Real Passband Signal

- take IFT to yield

$$x_{\mathrm{PB}}(t) = x_{\mathrm{BB}}(t)e^{2\pi i f_c t} + x^*_{\mathrm{BB}}(t)e^{-2\pi i f_c t} \tag{1}$$

$$= 2\ \mathrm{Re}\left\{x_{\mathrm{BB}}(t)e^{2\pi i f_c t}\right\} \tag{2}$$

$$= 2\ \mathrm{Re}\{x_{\mathrm{BB}}(t)\}\cos(2\pi f_c t) - 2\ \mathrm{Im}\{x_{\mathrm{BB}}(t)\}\sin(2\pi f_c t)$$

- define $x_{\mathrm{A}}(t) = x_{\mathrm{BB}}(t)e^{2\pi i f_c t}$, which is called the analytic representation of the passband signal $x_{\mathrm{PB}}(t)$

- thus,

$$x_{\mathrm{PB}}(t) = x_{\mathrm{A}}(t) + x^*_{\mathrm{A}}(t) \tag{3}$$

An Alternative Approach to Yield $x_{\mathrm{PB}}(t)$

- we have

$$\hat{x}_{\mathrm{PB}}(f) = \hat{x}_{\mathrm{PB}}\hat{x}_{\mathrm{H}}(f) + \hat{x}_{\mathrm{PB}}\hat{x}_{\mathrm{H}}(-f) \tag{4}$$

- where $\hat{x}_{\mathrm{H}}$ is called the Hilbert filter whose transfer function is a step function

- the IFT x_{H} of $\hat{x}_{\mathrm{H}}$ is given by

$$x_{\mathrm{H}}(t) = \frac{1}{2}\left[\delta(t) + j\frac{1}{\pi t}\right]$$

An Alternative Approach to Yield $x_{\mathrm{PB}}(t)$

- take IFT of (4) to yield

$$
\begin{aligned}
x_{\mathrm{PB}}(t) &= x_{\mathrm{PB}}(t) * x_{\mathrm{H}}(t) + x_{\mathrm{PB}}(t) * x_{\mathrm{H}}(-t) \qquad (5)\\
&= \frac{1}{2} x_{\mathrm{PB}}(t) * \left[\delta(t) + j\frac{1}{\pi t}\right] + \frac{1}{2} x_{\mathrm{PB}}(t) * \left[\delta(t) - j\frac{1}{\pi t}\right]\\
&= \frac{1}{2} x_{\mathrm{PB}}(t) + \frac{j}{2}\left[x_{\mathrm{PB}}(t) * \frac{1}{\pi t}\right]\\
&\qquad + \frac{1}{2} x_{\mathrm{PB}}(t) - \frac{j}{2}\left[x_{\mathrm{PB}}(t) * \frac{1}{\pi t}\right]\\
&= \frac{1}{2}\left[x_{\mathrm{PB}}(t) + j\ x_{\mathrm{PB}}(t) * \frac{1}{\pi t}\right]\\
&\qquad + \frac{1}{2}\left[x_{\mathrm{PB}}(t) - j\ x_{\mathrm{PB}}(t) * \frac{1}{\pi t}\right]
\end{aligned}
$$

An Alternative Approach to Yield $x_{\mathrm{PB}}(t)$

- this is the place where the Hilbert transform appears, i.e.,

$$x_{\mathrm{PB}}(t) = \frac{1}{2}\left[x_{\mathrm{PB}}(t) + j\ \tilde{x}_{\mathrm{PB}}(t)\right] + \frac{1}{2}\left[x_{\mathrm{PB}}(t) - j\ \tilde{x}_{\mathrm{PB}}(t)\right],$$

  where $\tilde{x}_{\mathrm{PB}}(t)$ is the Hilbert transform of $x_{\mathrm{PB}}(t)$, i.e., [3]

$$\begin{aligned}\tilde{x}_{\mathrm{PB}}(t) &= x_{\mathrm{PB}}(t) * \frac{1}{\pi t} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_{\mathrm{PB}}(\lambda)}{t-\lambda} d\lambda \end{aligned} \tag{6}$$

- note that we have $1/2\left[x_{\mathrm{PB}}(t) + j\ \tilde{x}_{\mathrm{PB}}(t)\right] = x_{\mathrm{A}}(t)$

[3] There is a singularity in the integrand of (6). Therefore, the meaning of (6) is made precise by defining the integral as the Cauchy principle value.

Recovering $x_{\mathrm{PB}}(t)$ from $x_{\mathrm{A}}(t)$

- in frequency domain

$$\hat{x}_{\mathrm{PB}}(f) = \hat{x}_{\mathrm{A}}(f) + \hat{x}^*_{\mathrm{A}}(-f) \tag{7}$$

- in time domain

$$x_{\mathrm{PB}}(t) = 2\ \mathrm{Re}\left(x_{\mathrm{A}}(t)\right) \tag{8}$$

Relating $\langle x_{\mathrm{PB}}, y_{\mathrm{PB}} \rangle$ to $\langle x_{\mathrm{A}}, y_{\mathrm{A}} \rangle$

- inner product

$$\langle x_{\mathrm{PB}}, y_{\mathrm{PB}} \rangle = 2 \ \mathrm{Re} \left(\langle x_{\mathrm{A}}, y_{\mathrm{A}} \rangle \right)$$

- energy

$$||x_{\mathrm{PB}}(t)||^2 = 2 \ ||x_{\mathrm{A}}(t)||^2$$

- verify the results above

Baseband Representation of Passband Signals

Baseband Representation

- we denote the baseband (BB) representation of $x_{\mathrm{PB}}(t)$ by $x_{\mathrm{BB}}(t)$
- we use BB representation, yet one cannot recover $x_{\mathrm{PB}}(t)$ from $x_{\mathrm{BB}}(t)$ alone without the carrier frequency f_c
- $x_{\mathrm{BB}}(t)$ enables certain operations to be performed and results to be derived without depending on f_c
- for engineers, $x_{\mathrm{BB}}(t)$ is the everything about $x_{\mathrm{PB}}(t)$

Baseband Representation



- BB representation of a real passband signal $x_{\mathrm{PB}}(t)$ that is bandlimited to B Hz around the carrier frequency f_c is the complex signal defined as

$$x_{\mathrm{BB}}(t) = e^{-i2\pi f_c t} x_{\mathrm{A}}(t), \tag{9}$$

  where $x_{\mathrm{A}}(t)$ is the analytic representation of $x_{\mathrm{PB}}(t)$

- note that
  - $|x_{\mathrm{BB}}(t)| = |x_{\mathrm{A}}(t)|$
  - $\hat{x}_{\mathrm{BB}}(f) = \hat{x}_{\mathrm{A}}(f + f_c)$
- illustrate the idea!

From x_{PB} to x_{BB}

- x_{BB} is also given by

$$x_{\mathrm{BB}}(t) = e^{-i2\pi f_c t} x_{\mathrm{PB}}(t) * \mathtt{LPF}_{W_c}(t) \tag{10}$$

- here $\mathtt{LPF}_{W_c}(t) = 2W_c \ \mathrm{sinc}(2W_c t)$
- W_c is arbitrarily chosen in the range $(W/2) \leq W_c \leq 2f_c - (W/2)$
- illustrate the idea!

The In-Phase and Quadrature Components

Convolution and Its Real Part

- real part of the convolution $\neq$ the convolution of their real parts
- but if one signal is real, then things become simpler
- i.e., for complex $x(t)$ and real $h(t)$, we have

$$\mathrm{Re}\{x * h\} = \mathrm{Re}\{x\} * h \tag{11}$$

$$\mathrm{Im}\{x * h\} = \mathrm{Im}\{x\} * h \tag{12}$$

- let us apply above result to (10)

Convolution and Its Real Part

- applying (11) and (12) into (10), yields

$$\text{Re}\{x_{\text{BB}}(t)\} = x_{\text{PB}}(t)\cos(2\pi f_c t) * \text{LPF}_{W_c}(t) \tag{13}$$

$$\text{Im}\{x_{\text{BB}}(t)\} = -x_{\text{PB}}(t)\sin(2\pi f_c t) * \text{LPF}_{W_c}(t) \tag{14}$$

- engineering literature:
  - $\text{Re}\{x_{\text{BB}}(t)\} \rightarrow$ in-phase component of x_{PB}
  - $\text{Im}\{x_{\text{BB}}(t)\} \rightarrow$ quadrature component of x_{PB}
- draw a diagram to yield in-phase and quadrature components from a given real passband signal

Obtaining the BB Representation

Recovering $x_{\mathrm{PB}}(t)$ from $x_{\mathrm{BB}}(t)$

- simply follows from (7), (8), and (9)
- in frequency domain

$$\hat{x}_{\mathrm{PB}}(f) = \hat{x}_{\mathrm{BB}}(f - f_c) + \hat{x}^*_{\mathrm{BB}}(-f - f_c)$$

- in time domain

$$\begin{aligned} x_{\mathrm{PB}}(t) &= 2\ \mathrm{Re}\left\{x_{\mathrm{BB}}(t)e^{i2\pi f_c t}\right\} \\ &= 2\ \mathrm{Re}\{x_{\mathrm{BB}}(t)\}\cos(2\pi f_c t) - 2\ \mathrm{Im}\{x_{\mathrm{BB}}(t)\}\sin(2\pi f_c t) \end{aligned}$$

- illustrate!

Relating $\langle x_{\mathrm{PB}}, y_{\mathrm{PB}} \rangle$ to $\langle x_{\mathrm{BB}}, y_{\mathrm{BB}} \rangle$

- inner product

$$\langle x_{\mathrm{PB}}, y_{\mathrm{PB}} \rangle = 2 \ \mathrm{Re} \left\{ \langle x_{\mathrm{BB}}, y_{\mathrm{BB}} \rangle \right\}$$

- energy

$$||x_{\mathrm{PB}}(t)||^2 = 2 \ ||x_{\mathrm{BB}}(t)||^2$$

- verify the results above [4]

[4] Start by showing that $\langle x_{\mathrm{BB}}, y_{\mathrm{BB}} \rangle = \langle x_{\mathrm{A}}, y_{\mathrm{A}} \rangle$. Then use the results in page 17.

Bandwidth of x_{PB} and x_{BB}

- real passband signal x_{PB} is of bandwidth W Hz $\rightarrow$ x_{BB} is a signal of bandwidth $W/2$ Hz

BB representation of $x_{\mathrm{PB}} * y_{\mathrm{PB}}$

- x_{PB} is real passband, bandlimited to W Hz around f_c, y_{PB} also have similarly properties
- then BB representation of $(x_{\mathrm{PB}} * y_{\mathrm{PB}})$ is $(x_{\mathrm{BB}} * y_{\mathrm{BB}})$ [5]
- illustrate the idea!

[5] Additional assumptions are to be made to be mathematically accurate, even though we have avoided those to make the big picture clear, e.g., defining the membership of signals in $\mathcal{L}_1$, or $\mathcal{L}_2$.

BB representation of $x_{\mathrm{PB}} * h$

- x_{PB} is real passband, bandlimited to W Hz around f_c, h is a real stable filter
- then BB representation of $(x_{\mathrm{PB}} * h)$ is the IFT of

$$\hat{x}_{\mathrm{BB}}(f)\hat{h}(f + f_c)$$

- i.e.,

$$(x_{\mathrm{PB}} * h)_{\mathrm{BB}}(t) = \int_{-\infty}^{\infty} \hat{x}_{\mathrm{BB}}(f)\hat{h}(f + f_c)e^{i2\pi ft}df$$

- illustrate the idea!

Recall: if the Baseband Signal is Real?

- the resulting passband waveform $x(t)$ is

$$\begin{aligned} x(t) &= u(t)e^{2\pi i f_c t} + u(t)e^{-2\pi i f_c t} \\ &= 2\ u(t)\cos(2\pi f_c t) \end{aligned}$$

- since $u(t)$ is real, these two bands are redundant and either could be reconstructed from the other

- thus, the above passband signal is quite wasteful: half of the band is redundant

Recall: Complex Baseband Signal

- can we do better? yes, how?
- make the baseband signal not conjugate-symmetric
- i.e., make the baseband signal is complex

Recall: Complex Baseband Signal

- can we do better? yes, how?
- make the baseband signal not conjugate-symmetric
- i.e., make the baseband signal is complex
- the simplest and most widely employed solution for using this wasted bandwidth is quadrature amplitude modulation (QAM)

Appendix

Conjugate Symmetry

- if a waveform $x(t)$ is real, its FT $\hat{x}(f)$ is conjugate-symmetric
- i.e., $\hat{x}(-f) = \hat{x}^*(f)$, and thus
  - $|\hat{x}(f)| = |\hat{x}(-f)|$, magnitude of $\hat{x}$ is even-symmetric
  - the phase of $\hat{x}$, denoted $\theta(f)$ is odd-symmetric

Some Useful FT pairs

$$x(t) \longleftrightarrow \hat{x}(f) \tag{15}$$

$$\hat{x}(t) \longleftrightarrow x(-f) \tag{16}$$

$$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|}\, \hat{x}\left(\frac{f}{\alpha}\right), \quad \alpha \in \mathbb{R} \setminus \{0\} \tag{17}$$

$$x(-t) \longleftrightarrow \hat{x}\,(-f) \tag{18}$$

$$x^*(-t) \longleftrightarrow \hat{x}^*(f) \tag{19}$$

$$\mathsf{sgn}(t) \longleftrightarrow \frac{1}{i\pi f} \tag{20}$$

$$u(t) \longleftrightarrow \frac{1}{2}\, \delta(f) + \frac{1}{i2\pi f} \tag{21}$$