Digital Communications
EN5601

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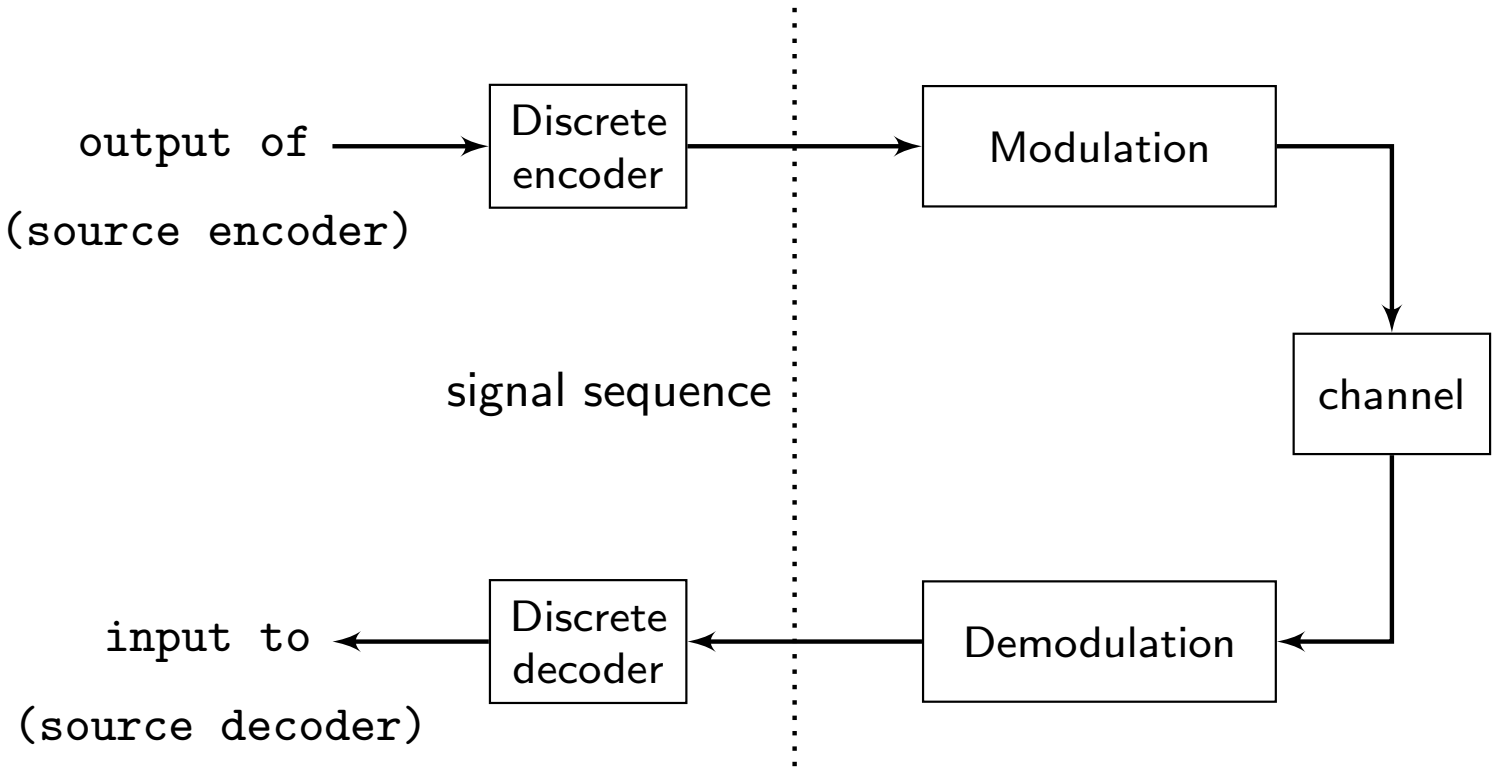
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Lecture 5: Channels, Modulation, and Demodulation (part I)

References

- the discussion is essentially based on [§ 6, Gallager-2008]

Channel Encoding / Decoding



Digital Modulation/ Demodulation

- digital modulation (channel encoding): the process of converting an input sequence of bits into a waveform suitable for transmission over a communication channel
- demodulation (channel decoding): the process at the receiver of converting the received waveform into a (perhaps noisy) replica of the input bit sequence

Recall..

- the source and channel waveforms of interest are real/complex $\mathcal{L}_2$ vectors
- any such vector x corresponds to an $\mathcal{L}_2$ signal $x(t)$ [1]
- we have

$$x(t) = \sum_k x_k \phi_k(t) \tag{1}$$

- R.H.S converges in $\mathcal{L}_2$ sense
- any sequence with $\sum_k |x_k|^2 < \infty$, $\exists$ an $\mathcal{L}_2$ function $x(t)$ satisfying (1)
- going back and forth is possible

[1] In fact, there is an equivalent class of $\mathcal{L}_2$ functions.

Choose the Basis

- $\{\phi_k\}_k$ is not necessarily a basis for $\mathcal{L}_2$
- it is a basis for some subspace of $\mathcal{L}_2$
- sometimes, it can be a sequence that are not orthonormal

Basic Parts of a Modulator

- sequence of binary digits $\rightarrow$ sequence of real/complex numbers $\rightarrow$ a waveforms
- a major objective and constraints when designing the modulator:
  - maximize the rate at which bits enter the encoder
  - guarantee a suitably small error rate
  - constraints on the transmitted power and bandwidth
  - constraints complexity, compatibility with standards

Signal Constellation

- consider the following transmitted waveform:

$$u(t) = \sum_k u_k \text{ sinc} \left(\frac{t}{T} - k \right) \tag{2}$$

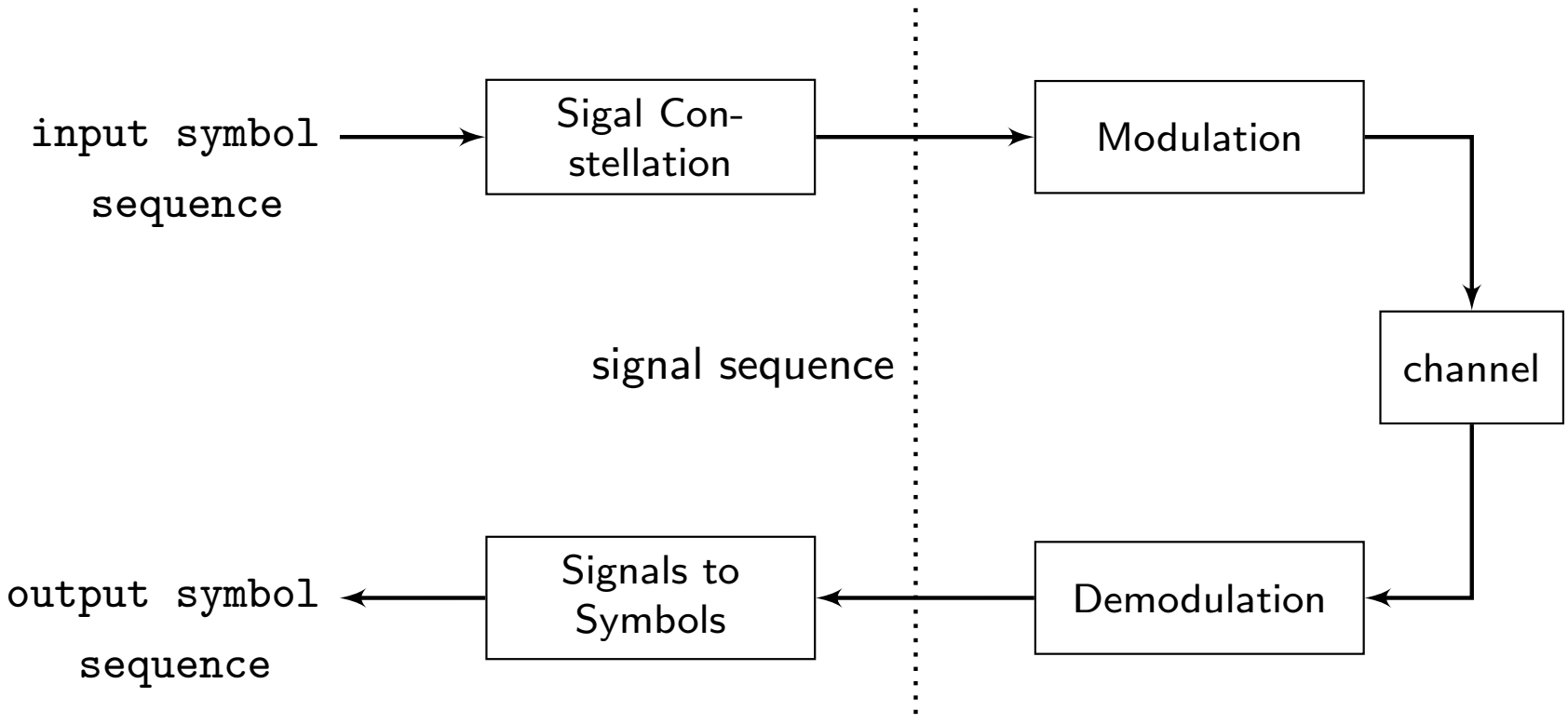
- sequence of binary digits $\rightarrow$ sequence of numbers of $\pm 1 \rightarrow$ a waveforms of the form (2)

- the T-spaced sinc in (2) can be replaced by any other set of orthogonal functions

Signal Constellation

- the rudimentary mapping $0 \rightarrow +1$ and $1 \rightarrow -1$ can also be replaced
  - segmenting the binary stream into b-tuples of binary digits (symbol sequence)
  - then map b-tuples into n-tuples of real or complex numbers (signal sequence)
  - the set of 2^b possible n-tuples resulting from this mapping is called a signal constellation

Channel Encoding / Decoding



Modulation Block

- modulation block usually has 2 layers: signals to baseband waveforms $\rightarrow$ baseband to passband
- let the baseband waveform be $u(t)$
- then the passband waveform is $x(t) = \text{Re}\left\{u(t)e^{2\pi i f_c t}\right\}$

PAM

Pulse Amplitude Modulation (PAM)

- one of the simplest type of modulation
- incoming bits, R b/s $\rightarrow$ symbols of b-bits each, R/b sym/s $\rightarrow$ signals (constellation), R/b signals/s $\rightarrow$ PM waveform
- PAM waveform $u(t)$ is given by

$$u(t) = \sum_k u_k p(t - kT), \tag{3}$$

where $p(t)$ is a basic pulse which is $\mathcal{L}_2$ and $T = 1/R_s = b/R$

Standard PAM Signal Constellation

- here the constellation is $\mathcal{A} = \{a_1, a_2, \ldots, a_M\}$, where $M = 2^b$
- in particular,

$$\mathcal{A} = \left\{ \frac{-d(M-1)}{2}, \frac{-d(M-3)}{2}, \cdots, \frac{d(M-3)}{2}, \frac{d(M-1)}{2} \right\}$$

Energy Per Signal

- note that $u_k \in \mathcal{A}$ for all k
- each u_k is a sample value of a RV U_k (assumed equiprobable over $\mathcal{A}$)
- moreover, the sequence U_k is IID

Energy Per Signal

- thus, the mean squared signal value (energy per signal) E_s is given by

$$E_s = \mathsf{E}[U_k^2] = \frac{d^2(M^2-1)}{12} = \frac{d^2(2^{2b}-1)}{12} \tag{4}$$

- unit increase in b increases E_s by a factor of 4
- thus, increasing the rate R by increasing b requires impractically large energy for large b

Received PAM Waveform

- attenuation and delay are largely separable from modulation
- recall that the baseband waveform $u(t)$ is converted to passband and transmitted
- at the receiver: conversion back to baseband
- the received waveform:

$$v(t) = u(t) + z(t)$$

- $z(t)$ is noise: determines the allowable values of d, affect the constellation points at Rx

Received PAM Waveform

- thus, we have at Rx

$$v(t) = \sum_k u_k p(t - kT) + z(t)$$

- let us, for now, ignore the noise and check how this simplest modulation type can choose the basic pulse $p(t)$

- compromise between time and bandwidth, no ISI

Pulses for PAM

Basic Pulse $p(t)$ for PAM

- we want to retrieve the coefficients u_k perfectly from $u(t)$
- PAM demodulator is a filter with response $q(t)$ $(\mathcal{L}_2)$ followed by a sampler
- received filtered waveform is given by

$$r(t) = \int_{-\infty}^{\infty} u(\tau) q(t-\tau) d\tau$$

- samples are $r(T), r(2T), \ldots$

Basic Pulse $p(t)$ for PAM

- why we impose some restrictions?
  - why one filter followed by a sampler?
  - why we dropped noise term and analyse?
  - because nothing to loose
- objective: determine $p(t)$ and $q(t)$ such that $r(kT) = u_k$ for all k
- check that $p(t) = (1/\sqrt{T})\text{sinc}(t/T)$ is a such choice
- note: $\text{sinc}(\cdot)$ has ∞ delay

Basic Pulse $p(t)$ for PAM

- thus the received waveform $r(t)$ is as follows:

$$
\begin{aligned}
r(t) &= \int_{-\infty}^{\infty} \sum_k u_k p(\tau - kT) q(t-\tau) d\tau \\
&= \sum_k u_k \int_{-\infty}^{\infty} p(\tau - kT) q(t-\tau) d\tau \\
&= \sum_k u_k \underbrace{\int_{-\infty}^{\infty} p(\lambda) q\left((t-kT) - \lambda\right) d\tau}_{= (p*q)(t)|_{t=t-kT}} \\
&= \sum_k u_k g(t-kT),
\end{aligned}
$$

where $g = p * q$ [2]

[2] The waveform $g(t)$ is assumed to be an $\mathcal{L}_2$ function, though this is not true in general.

Basic Pulse $p(t)$ for PAM

- only the convolution g that determines r
- when channel noise is added, the individual choice of p and q matter

Ideal Nyquist Property

Ideal Nyquist Waveforms

- recall the objective.. $r(kT) = u_k$ for all k
- i.e.,

$$r(kT) = \sum_j u_j g\left((k-j)T\right) = u_k, \ \forall \ k$$

- thus, ideal Nyquist property is

$$g(0) = 1 \text{ and } g(k) = 0 \text{ for all } k \neq 0 \tag{5}$$

- such waveforms: ideal Nyquist with interval T

Ideal Nyquist Waveforms

- rect(t/T): not band-limited
- sinc(t/T): not time-limited
- we want to find functions ideal Nyquist waveforms that are approximately baseband-limited and approximately time-limited

Nyquist Criterion

A Necessary and Sufficient Condition

- lets derive a necessary and sufficient condition for ideal Nyquist property
- let $s(t)$ defined as

$$s(t) = \sum_k g(kT)\text{sinc}\left(\frac{t}{T} - k\right) \tag{6}$$

- $g(t)$ is ideal Nyquist $\implies$ $s(t) = \text{sinc}(t/T)$
- $s(t) = \text{sinc}(t/T)$ $\implies$ $g(t)$ is ideal Nyquist
- i.e., a necessary and sufficient condition: $s(t) = \text{sinc}(t/T)$

A Necessary and Sufficient Condition

- the condition above can further be transformed into a property in frequency domain
- we have [3]

$$g(t) \text{ is ideal} \implies s(t) = \text{sinc}(t/T) \tag{7}$$

$$\text{Nyquist} \implies \text{sinc}(t/T) = \sum_k g(kT)\text{sinc}\left(\frac{t}{T} - k\right) \tag{8}$$

$$\implies T\text{rect}(fT) = \text{l.i.m} \sum_k \hat{g}(f + k/T)\text{rect}(fT) \tag{9}$$

where (8) follows from (6) and (9) follows from FT properties for $\mathcal{L}_2$ signals

[3] The FT used in relation (9) relies on the Aliasing Theorem, see § 4.7.4 of [Gallager-2008].

A Necessary and Sufficient Condition

- conversely,

$$T\text{rect}(fT) = \text{l.i.m} \sum_k \hat{g}\left(f + \frac{k}{T}\right) \text{rect}(fT) \implies \hat{s}(f) \overset{\mathcal{L}_2}{\approx} T\text{rect}(fT)$$

$$\implies s(t) = \text{sinc}(t/T)$$

$$\implies g(t) \text{ is Ideal Nyquist}$$

A Necessary and Sufficient Condition

- thus, the Nyquist criterion is

$$T \text{ rect}(fT) = \text{l.i.m} \sum_k \hat{g}\left(f + \frac{k}{T}\right) \text{rect}(fT)$$

- see Theorem 6.3.1 of [Gallager-2008]

A Necessary and Sufficient Condition

- thus, the Nyquist criterion is

$$T \text{ rect}(fT) = \text{l.i.m} \sum_k \hat{g}\left(f + \frac{k}{T}\right) \text{rect}(fT)$$

- the nominal or Nyquist bandwidth associated with a PAM pulse $g(t)$ with signal interval T is defined to be $W = 1/2T$

- see Theorem 6.3.1 of [Gallager-2008]

Ideal Nyquist Waveforms (Revisit)

- recall.. we want to find functions ideal Nyquist waveforms that are approximately baseband-limited and approximately time-limited
- but we seek cases that are strictly band-limited
- i.e., the waveforms we seek are not strictly time-limited
- frequency constraint is the more stringent
- time constraint is usually more flexible and can be imposed as an approximation

Band Edge Symmetry

- actual BW B of the pulse should be close to $W{=}1/(2T)$
- Discuss different signalling rates for a given bandwidth B for a baseband PAM waveform
  1. $W = B$
  2. $W < B < 2W$
  3. $B = 2W$
  4. $B > 2W$
  5. $B < W$
- reading: § 6.3.1 of [Gallager-2008]

Band Edge Symmetry

- if $\hat{g}(\cdot)$ is complex, $\mathrm{Re}\{\hat{g}(f)\}$ must satisfy the band edge symmetry to meet the Nyquist criterion
- choosing $\mathrm{Im}\{\hat{g}(f)\} \neq 0$ just more work with little effect
- so engineers happy with just real $\hat{g}(f)$, e.g., raised cosine pulses
- moreover, integrating noise in to the picture necessitate $|\hat{p}(f)| = |\hat{q}(f)|$
- $\hat{g}(f) = \hat{p}(f)\hat{q}(f)$, which is real $\implies$ $\hat{q}(f) = \hat{p}^*(f)$ $\implies$ $q(t) = p^*(-t)$

$\{p(t-kT)\}_k$ as an Orthonormal Basis

- the above result $q(t) = p(-t)$, together with that $g = p * q$ and the ideal Nyquist property [see (5)] yields

$$g(kT) = \int_{-\infty}^{\infty} p(\tau)p^*(\tau - kT)d\tau = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \ldots \end{cases}$$

- i.e., $\{p(t-kT)\}_{k\in\mathbf{Z}}$ is an orthonormal set of functions

$\{p(t-kT)\}_k$ as an Orthonormal Basis

- thus, the PAM transmitted waveform $u(t) = \sum_k u_k p(t-kT)$ is an orthonormal expansion
- retrieving the coefficient u_k corresponds to projecting $u(t)$ onto the 1D subspace spanned by $p(t-kT)$
- this projection is accomplished by filtering $u(t)$ by $q(t)$ and then sampling at time kT
- the filter $q(t)$ is called the matched filter to $p(t)$

$\{p(t-kT)\}_k$ as an Orthonormal Basis

- Theorem 6.3.2, [Gallager-2008]: Let $p(t)$ be an $\mathcal{L}_2$ function such that $\hat{g}(f) = |\hat{p}(f)|^2$ satisfies the Nyquist criterion for T. Then $\{p(t-kT)\}_{k\in\mathbf{Z}}$ is a set of orthonormal functions. Conversely, if $\{p(t-kT)\}_{k\in\mathbf{Z}}$ is a set of orthonormal functions, then $|\hat{p}(f)|^2$ satisfies the Nyquist criterion.