Digital Communications
EN5601

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Lecture 4: Random Processes and Noise

References

- the discussion is essentially based on [§ 7, Gallager-2008]

Random Processes

- A random process $\{Z(t)\}_{t\in\mathbb{R}}$ is a collection of random variables (RV), one for each $t \in \mathbb{R}$
- t usually models time, and any given instant in time is often referred to as an epoch
- the range of t can be restricted as well, e.g., $\{Z(t)\}_{t\in[a,b]}$

RVs from Random Processes

- there is one RV for each epoch t, denoted $Z(t)$
- there must be an underlying sample space Ω over which the RVs are defined [1]
- e.g., for each epoch $t \in \mathbb{R}$, the RV $Z(t)$ is a function $Z(t, \cdot)$ that maps sample points $\omega \in \Omega$ to real numbers

[1] You may refer to § 2.3 of [Wozencraft-1965] for a review of RVs.

Sample Functions from Random Processes

- for a given sample point $\omega \in \Omega$, consider the collection of all the sample values $\{Z(t, \omega)\}_{t \in \mathbb{R}}$
- the collection aforementioned is called a sample function $z(\cdot)$ of the process, where $z : \mathbb{R} \to \mathbb{R}$
- i.e., $z(\cdot)$ is a sample function of the random process $\{Z(t)\}_{t \in \mathbb{R}}$
- draw a picture to visualize RVs and sample functions of a random process

Examples 1

- let $\{Z_k\}_{k\in\mathbb{N}}$ be a sequence of RVs defined on some sample space Ω
- let $\{\phi_n(\cdot)\}_{n\in\mathbb{N}}$ be a sequence of orthonormal real functions
- for each $t \in \mathbb{R}$ define

$$Z(t) = \sum_{k\in\mathbb{N}} Z_k \phi_n(t) \tag{1}$$

- it is clear that random processes can be defined as above
- but it is less clear that this will provide a mechanism for constructing reasonable models of actual physical noise processes

Examples 2

- let $Z_1, Z_2, \ldots$ be a sequence of independent and identically distributed (IID) RVs which take the values ± 1 with probability $1/2$ each
- suppose the modulation pulse $\text{sinc}(t/T)$ is used to yield

$$Z(t) = \sum_{k \in \mathbb{N}} Z_k \ \text{sinc}\left(\frac{t - kT}{T}\right) \tag{2}$$

- at each epoch kT, the RV $Z(kT)$ is simply the binary RV Z_k

Examples 3

- let $\{Z_k\}_{k\in\mathbf{Z}}$ be a sequence of IID zero-mean Gaussian RVs of variance σ^2
- for each $t \in \mathbb{R}$ define

$$Z(t) = \sum_{k\in\mathbf{Z}} Z_k \text{ sinc}\left(\frac{t-kT}{T}\right) \tag{3}$$

- at each epoch kT, the RV $Z(kT)$ is the zero-mean Gaussian RV Z_k

Examples 4

- let Z be a zero-mean Gaussian RVs of variance 1
- for each $t \in [-1, 1]$ define

$$Z(t) = tZ \tag{4}$$

- sample functions are simply straight lines through the origin with random slope

Mean of a Random Process

- let $\{Z(t)\}_{t\in\mathbb{R}}$ be a random process
- the mean at each epoch t

$$\mathsf{E}\left[Z(t)\right] = \bar{Z}(t) \tag{5}$$

  is simply a real-valued function of t
- it is computed directly from the distribution function $F_{Z(t)}(\cdot)$ or density $f_{Z(t)}(\cdot)$
- compute $\bar{Z}(t)$ for above examples

Covariance of a Random Process

- let $\{Z(t)\}_{t\in\mathbb{R}}$ be a random process
- covariance $\mathtt{K}_Z(\cdot,\cdot)$ is a real-valued function of the epochs t and τ
- it is defined by

$$\mathtt{K}_Z(t,\tau) = \mathsf{E}\left[\left(Z(t)-\bar{Z}(t)\right)\left(Z(\tau)-\bar{Z}(\tau)\right)\right] \tag{6}$$

- it is found directly from the joint distribution function $F_{Z(t)Z(\tau)}(\cdot,\cdot)$ or density $f_{Z(t)Z(\tau)}(\cdot,\cdot)$
- computing $\mathtt{K}_Z(t,\tau)$ for above examples! not trivial

Covariance: for Example 1

- for simplicity, assume $Z_1, Z_2, \ldots$ are IID with variance σ^2
- let $\tilde{Z}(t) = Z(t) - \bar{Z}(t)$ denote the fluctuation of RV $Z(t)$
- then we have

$$
\begin{aligned}
\mathtt{K}_Z(t,\tau) &= \mathsf{E}\left[\left(Z(t) - \bar{Z}(t)\right)\left(Z(\tau) - \bar{Z}(\tau)\right)\right] \\
&= \mathsf{E}\left[\left(\textstyle\sum_{k\in\mathbf{N}} \tilde{Z}_k \phi_k(t)\right)\left(\sum_{m\in\mathbf{N}} \tilde{Z}_m \phi_m(\tau)\right)\right] \\
&= \mathsf{E}\left[\textstyle\sum_{k\in\mathbf{N}} \tilde{Z}_k^2 \phi_k(t)\phi_k(\tau) + \sum_{k\neq m} \tilde{Z}_k \tilde{Z}_m \phi_k(t)\phi_k(\tau)\right] \\
&= \textstyle\sum_{k\in\mathbf{N}} \underbrace{\mathsf{E}(\tilde{Z}_k^2)}_{=\sigma^2} \phi_k(t)\phi_k(\tau) + \sum_{k\neq m} \underbrace{\mathsf{E}(\tilde{Z}_k \tilde{Z}_m)}_{=0} \phi_k(t)\phi_k(\tau) \\
&= \sigma^2 \textstyle\sum_{k\in\mathbf{N}} \phi_k(t)\phi_k(\tau)
\end{aligned}
$$

Covariance: for Example 2

- it can be shown that

$$\begin{aligned} \mathtt{K}_Z(t,\tau) &= \sum_{k\in\mathbb{N}} \mathsf{sinc}\left(\frac{t-kT}{T}\right)\mathsf{sinc}\left(\frac{\tau-kT}{T}\right) \\ &= \mathsf{sinc}\left(\frac{t-\tau}{T}\right) \end{aligned}$$

Covariance: for Example 3

- it can be shown that

$$\begin{aligned} \mathtt{K}_Z(t,\tau) &= \sigma^2 \sum_{k\in\mathbb{N}} \mathsf{sinc}\left(\frac{t-kT}{T}\right) \mathsf{sinc}\left(\frac{\tau-kT}{T}\right) \\ &= \sigma^2 \mathsf{sinc}\left(\frac{t-\tau}{T}\right) \end{aligned}$$

Define Gaussian Processes

- definition 7.2.3 of [Gallager-2008] [2]
- $\{Z(t)\}_{t\in\mathbb{R}}$ is a zero mean Gaussian process if, for all positive integers n and all finite sets of epochs $t_1, \ldots, t_n$, the set of random variables $Z(t_1), \ldots, Z(t_n)$ is a (zero-mean) jointly Gaussian set of RVs

[2] You may also refer to § 3.4 of [Wozencraft-1965].

Define Gaussian Processes

- theorem 7.3.2 of [Gallager-2008]
- a zero mean Gaussian process is specified by its covariance function $\mathtt{K}(\cdot, \cdot)$
- how ?

Linear Functional for Random Processes

- suppose the sample function $Z(\cdot, \omega)$ of random processes $\{Z(t)\}_{t \in \mathbb{R}}$ are real $\mathcal{L}_2$ waveforms
- i.e., sample functions can be viewed as vectors in the $\mathcal{L}_2$ inner product space
- let $g(\cdot)$ be a real $\mathcal{L}_2$ function

Linear Functional for Random Processes

- then the following inner product is defined: [3]

$$\langle Z(\cdot,\omega), g(\cdot)\rangle = \int_{-\infty}^{\infty} Z(t,\omega)g(t)dt$$

- $\langle Z(\cdot,\omega)g(\cdot)\rangle$ is the mapping of sample point ω into a real number

- thus $V(\omega) = \int_{-\infty}^{\infty} Z(t,\omega)g(t)dt$ is the mapping of sample point ω into a real numbers $\rightarrow V$ is a random variable

[3] The inner product below is finite (apply Schwarz inequality).

Linear Functional for Random Processes

- V above is called the linear functional of the process $\{Z(t)\}_{t\in\mathbb{R}}$
- V is denoted by

$$V = \int_{-\infty}^{\infty} Z(t)g(t)dt$$

Linear Functional for Random Processes

- example
  - demodulator of PAM and QAM contain a filter $q(t)$ followed by a sampler
  - for an input $U(t)$ the output of the filter at a sampling time kT is

$$V = \int U(t)q(kT - t)dt$$

    which is a RV

Filters for Random Processes

- suppose the sample function $Z(\cdot, \omega)$ of random processes $\{Z(t)\}_{t \in \mathbb{R}}$ are real $\mathcal{L}_2$ waveforms
- let $h(\cdot)$ be the impulse response of a filter, a real $\mathcal{L}_2$ function
- then for any $\mathcal{L}_2$ sample function $Z(\cdot, \omega)$, the filter output at time τ is the inner product

$$\langle Z(\cdot, \omega) h_\tau(\cdot) \rangle = \int_{-\infty}^{\infty} Z(t, \omega) h(\tau - t) dt$$

where $h_\tau(t) = h(\tau - t)$

Filters for Random Processes

- for each τ, $\langle Z(\cdot,\omega)h_\tau(\cdot)\rangle$ is the mapping of sample point $\omega \in \Omega$ into a real number
- thus, $V(\tau,\omega) = \int Z(t,\omega)h(\tau - t)dt$ is the mapping of sample point $\omega \in \Omega$ into a real number
- for each τ, $V(\tau,\cdot)$ maps sample points $\omega \in \Omega$ into real numbers

Filters for Random Processes

- there is one RV for each epoch τ, denoted $V(\tau)$
- this means that $\{V(\tau)\}_{\tau \in \mathbb{R}}$ is a random process
- $V(\tau)$ above is called the filtered process resulting from passing $Z(t)$ through the filter $h(t)$
- $V(\tau)$ is denoted by

$$V(\tau) = \int_{-\infty}^{\infty} Z(t)h(\tau - t)dt$$

Gaussian Processes via Orthonormal Expansions

Gaussian Processes: Orthorn. Expansions

- zero-mean Gaussian process can be defined as follows:

$$Z(t) = \sum_{k \in \mathbb{N}} Z_k \phi_k(t), \tag{7}$$

where

  - $\{\phi_k(\cdot)\}_{k \in \mathbb{N}}$ is a sequence of real orthonormal functions
  - $\{Z_k\}_{k \in \mathbb{N}}$ is a sequence of if independent Gaussian RVs each with variance σ_k^2

Gaussian Processes via Orthonormal Expansions
Case 1: $\sigma_k^2 = 0$ for $k > n$

Jointly Gaussian Property

- in (7), suppose $\sigma_k^2 = 0$ for $k > n$
- thus we have

$$Z(t) = \sum_{k=1}^{n} Z_k \phi_k(t)$$

- verify that $V = (Z(t_1), \ldots, Z(t_l))$ is jointly Gaussian for all $l, t_1, \ldots, t_l$
- this verifies that $\{Z(t)\}_{t \in \mathbb{R}}$ is a Gaussian random process

Energy and Covariance Function

- energy in any sample function $z(\cdot)$ defined by [4]
  $z(t) = \sum_{k=1}^{n} z_k \phi_k(t)$ is $\sum_{k=1}^{n} z_k^2$ which is finite
- i.e., all the sample functions are $\mathcal{L}_2$
- verify that the covariance function $\mathtt{K}(\cdot, \cdot)$ is

$$\mathtt{K}(t, \tau) = \sum_{k,m} \mathsf{E}[Z_k Z_m] \phi_k(t) \phi_m(\tau) = \sum_{k=1}^{n} \sigma_k^2 \phi_k(t) \phi_m(\tau)$$

[4] When sample function is specified, sample point $\omega \in \Omega$ is also specified. Thus, $z(t) = Z(t, \omega) = \sum Z_k(\omega) \phi_k(t) = \sum z_k \phi_k(t)$

Linear Functional

- suppose $\omega \in \Omega$ is specified
- the linear functional $\langle Z(\cdot, \omega) g(\cdot) \rangle$, where $g(\cdot)$ is an $\mathcal{L}_2$ function, is given by

$$
\begin{aligned}
V(\omega) &= \int_{-\infty}^{\infty} Z(t, \omega) g(t) dt = \int_{-\infty}^{\infty} \sum_{k=1}^{n} Z_k(\omega) \phi_k(t) g(t) dt \\
&= \sum_{k=1}^{n} \int_{-\infty}^{\infty} Z_k(\omega) \phi_k(t) g(t) dt \\
&= \sum_{k=1}^{n} Z_k(\omega) \underbrace{\int_{-\infty}^{\infty} \phi_k(t) g(t) dt}_{\alpha_k < \infty} \\
&= \sum_{k=1}^{n} \alpha_k Z_k(\omega)
\end{aligned}
$$

- thus, $V = \sum_{k=1}^{n} \alpha_k Z_k$

Linear Functional

- V is a weighted sum of the zero-mean independent Gaussian RVs
- thus, V is zero-mean Gaussian with variance

$$\sigma_V^2 = \sum_{k=1}^{n} \alpha_k^2 \sigma_k^2$$

Linear Functional

- the linear functional $\langle Z(\cdot,\omega)g_m(\cdot)\rangle$, where $g_m(\cdot)$ is an $\mathcal{L}_2$ function, is given by

$$V_m(\omega) = \sum_{k=1}^{n} Z_k(\omega) \underbrace{\int_{-\infty}^{\infty} \phi_k(t) g_m(t) dt}_{\alpha_k^{(m)} < \infty}$$

$$= \sum_{k=1}^{n} \alpha_k^{(m)} Z_k(\omega)$$

- thus, $V_m = \sum_{k=1}^{n} \alpha_k^{(m)} Z_k$ is Gaussian
- if a sequence $\{g_m(t)\}_{1\le m\le l}$ of $\mathcal{L}_2$ functions is given, then we have $V = (V_1, \ldots, V_l)$ jointly Gaussian

Linear Filtering of the Process

- check it by your self
- find the filtered process resulting from passing $Z(t)$ through the filter $h(t) \in \mathcal{L}_2$
- in particular, show that

$$\langle Z(\cdot,\omega)h_\tau(\cdot)\rangle = \int_{-\infty}^{\infty} Z(t,\omega)h(\tau-t)dt = \sum_{k=1}^{n} Z_k(\omega)\langle\phi_k(\cdot)h_\tau(\cdot)\rangle$$

- show that all sample functions of the filtered process are $\mathcal{L}_2$
- show that the filtered process is a zero-mean Gaussian process

Gaussian Processes via Orthonormal Expansions
Case 1: $\sum_{k \in \mathbf{N}} \sigma_k^2 < \infty$

Gaussian Processes: Properties

- in (7), suppose $\sum_{k\in\mathbb{N}} \sigma_k^2 < \infty$
- roughly speaking all the properties discussed above are still valid
- however, some of the properties hold with probability 1
- more careful attention is to be made when deriving the results

Gaussian Processes: Properties

- Theorem 7.4.1 of [Gallager-2008]: Let $\{Z(t)\}_{t\in\mathbb{R}}$ be a Gaussian process $Z(t)=\sum_{k\in\mathbb{N}} Z_k\phi_k(t)$, where $\{Z_k\}_{k\in\mathbb{N}}$ is a sequence of independent Gaussian RVs $\mathcal{N}(0,\sigma^2)$, where $\sum_{k\in\mathbb{N}}\sigma^2<\infty$ and $\{\phi_k(\cdot)\}_{k\in\mathbb{N}}$ is an orthonormal set. Then
  - for any set of $\mathcal{L}_2$ waveforms $g_1(\cdot),\ldots,g_l(\cdot)$, the linear functionals $V_m,\ m\in\{1,\ldots,l\}$ given by $V_m=\int_\infty^\infty Z(t)g_m(t)dt$ are jointly Gaussian;
  - for any filter with real $\mathcal{L}_1$ and $\mathcal{L}_2$ impulse response $h(t)$, the filter output $\{V(\tau)\}_{\tau\in\mathbb{R}}$ given by

$$V(\tau)=\int_{-\infty}^{\infty} Z(t)h(\tau-t)dt=\sum_{k\in\mathbb{N}} Z_k\int_{-\infty}^{\infty}\phi_k(t)h(\tau-t)dt$$

    is a zero-mean Gaussian process.

Covariance for Linear Functionals

Covariance for Linear Functionals

- assume $\{Z(t)\}_{t\in\mathbb{R}}$ is a random process with zero mean
- $g_m(\cdot)$, $m \in \{1, \ldots, l\}$ are real $\mathcal{L}_2$ functions
- recall the linear functional $V_m = \int_{-\infty}^{\infty} Z(t)g_m(t)dt$
- then we have

$$\begin{aligned}
\mathsf{E}[V_j V_m] &= \mathsf{E}\left[\int_{-\infty}^{\infty} Z(t)g_j(t)dt \int_{-\infty}^{\infty} Z(\tau)g_m(\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(t)\mathsf{E}\left[Z(t)Z(\tau)\right] g_m(\tau)dtd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(t)\mathtt{K}_Z(t,\tau)g_m(\tau)dtd\tau
\end{aligned}$$

Covariance for Filtered Output

Covariance for Filtered Output

- assume $\{Z(t)\}_{t\in\mathbb{R}}$ is a random process with zero mean
- $h(\cdot)$ be a real $\mathcal{L}_2$ function, the impulse response
- recall the convolution $V(\tau) = \int_{-\infty}^{\infty} Z(t)h(\tau - t)dt$
- thus the covariance of the output

$$\begin{aligned}
\mathtt{K}_V(r,s) &= \mathsf{E}[V(r)V(s)] \\
&= \mathsf{E}\left[\int_{-\infty}^{\infty} Z(t)h(r-t)dt \int_{-\infty}^{\infty} Z(\tau)h(s-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(r-t)\mathtt{K}_Z(t,\tau)h(s-\tau)dtd\tau
\end{aligned}$$

Stationarity and Related Concepts

Stationary Random Processes

- stationary random processes have the property that the location of the time origin is irrelevant
- i.e. they behave the same way at one time as at any other time
- you should have already heard those definitions: see Definition 7.5.1 of [Gallager-2008]

Stationarity for Gaussian Processes

- Thorem 7.5.1 of [Gallager-2008]: A zero-mean Gaussian process $\{Z(t)\}_{t\in\mathbb{R}}$ is stationary if and only if

$$\mathtt{K}_Z(t_1,t_2)=\mathtt{K}_Z(t_1-t_2,0) \text{ for all } t_1,t_2\in\mathbb{R}.$$

- a nonzero mean Gaussian process is stationary if and only if its mean $\bar{Z}$ is constant and

$$\mathtt{K}_Z(t_1,t_2)=\mathtt{K}_Z(t_1-t_2,0) \text{ for all } t_1,t_2\in\mathbb{R}.$$

Wide-Sense Stationarity Random Processes

- in short, WSS random processes
- a random process $\{Z(t)\}_{t\in\mathbb{R}}$ is WSS if $\mathsf{E}[Z(t_1)] = \mathsf{E}[Z(0)]$ and
$$\mathtt{K}_Z(t_1, t_2) = \mathtt{K}_Z(t_1 - t_2, 0) \text{ for all } t_1, t_2 \in \mathbb{R}$$
- for WSS processes covariance function is a function of only one variable
- thus, we use $\tilde{\mathtt{K}}_Z(\tau)$ in place of $\mathtt{K}_Z(t+\tau, t)$ is

Stationarity in Practice

- stationary noise models are simple to work with
- but the sample functions of these processes do not fit into the $\mathcal{L}_2$ theoryof waveforms
- e.g., Example 3 in page 8: sample functions of this process have infinite energy with probability 1
- one practical solution to this is to simply truncate the process in any convenient way
- thus, when we say that a process is stationary, we mean that it is stationary within a much longer time interval than the interval of interest for communication

White Gaussian Noise

White Gaussian Noise (WGN)

- physical noise processes are very often reasonably modeled as zero-mean, stationary, and Gaussian
- further simplified assuming that the covariance between the noise at two epochs dies out very rapidly as the interval between those epochs increases
- this means that the covariance function $\tilde{\mathrm{K}}_Z(\tau)$ looks like a short-duration pulse around $\tau = 0$
- we refer to a zero-mean WSS Gaussian random process with such a narrow covariance function as white Gaussian noise (WGN)

White Gaussian Noise

- thus the covariance of linear functionals becomes [5]

$$
\begin{aligned}
\mathsf{E}[V_j V_m] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(t)\mathtt{K}_Z(t,\tau)g_m(\tau)dtd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(t)\mathtt{K}_Z(t-\tau,0)g_m(\tau)dtd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(t)\tilde{\mathtt{K}}_Z(t-\tau)g_m(\tau)dtd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(t)\frac{N_0}{2}\delta(t-\tau)g_m(\tau)dtd\tau \\
&= (N_0/2)\int_{-\infty}^{\infty} g_j(t)g_m(t)dt
\end{aligned}
$$

[5] The intensity or the spectral density of the WGN is denoted by the symbol $N_0/2$.

White Gaussian Noise

- we can also note that $\mathsf{E}[V_m^2] = (N_0/2) \int_{-\infty}^{\infty} g_m^2(t)dt$
- assuming $g_m(\cdot)$s are orthonormal, we have

$$\mathsf{E}[V_j V_m] = (N_0/2)\delta_{jm}$$

- reading page 245-246 of [Gallager-2008],stuff related to WGN

Appendix

Gaussian Random Variable

- a RV W is a Gaussian RV, if it has the probability density

$$f_W(w) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(w-\bar{W})^2}{2\sigma^2}\right), \tag{8}$$

- mean of W is $\mathsf{E}[W] = \bar{W}$
- variance of W is $\mathsf{E}[(W-\bar{W})^2] = \sigma^2$

Normalized Gaussian Random Variable

- a RV W is a normalized Gaussian RV, or more briefly a normal RV, if $\bar{W} = 0$ and $\sigma^2 = 1$
- its probability density is

$$f_W(w) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-w^2}{2}\right) \tag{9}$$

Mean and Covariance of a Random Vector

- let $W = (W_1, \ldots, W_n)$ be an n-tuple of RVs
- mean of W, denoted $\bar{W}$ is given by $\bar{W} = (\bar{W}_1, \ldots, \bar{W}_n)$
- the covariance matrix of W, denoted $\mathtt{K}_W$ is given by

$$\mathtt{K}_W = \mathsf{E}\left[\left(W - \bar{W}\right)\left(W - \bar{W}\right)^{\mathsf{T}}\right]$$

- note that
$[\mathtt{k}_W]_{mn} = \mathsf{E}\left[\left(W_m - \bar{W}_m\right)\left(W_n - \bar{W}_n\right)\right] = \mathsf{cov}(W_m, W_n)$

Jointly Gaussian Random Vector

- let $W_1, \ldots, W_l$ be IID normal RVs
- denote by W the vector of above IID RVs, i.e., $W = (W_1, \ldots, W_l)$
- denote by $\bar{Z}$ the vector $\bar{Z} = (\bar{Z}_1, \ldots, \bar{Z}_n)$
- then vector $Z \in \mathbb{R}^n$ defined by

$$Z = AW + \bar{Z} \tag{10}$$

is said to be jointly Gaussian, where $A \in \mathbb{R}^{n \times l}$

Jointly Gaussian Random Vector

- let

$$Z = AW + \bar{Z} \tag{11}$$

  which is defined as above

- $\mathsf{E}[Z] = \bar{Z}$
- $\mathtt{K}_Z = AA^{\mathsf{T}}$

Jointly Gaussian Random Vector

- probability density $f_Z(z)$ is given by [6]

$$f_Z(z) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\mathtt{K}_Z)}} \exp\left(-\frac{1}{2}(z-\bar{Z})^{\mathsf{T}}\mathtt{K}_Z^{-1}(z-\bar{Z})\right)$$

- recall: $W_1, \ldots, W_l$ are IID normal RVs, thus

$$f_W(w) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}w^{\mathsf{T}}w\right)$$

[6] If the covariance matrix is rank difficient, the Dirac delta function is to be used appropriately when determining the probability density function.