Digital Communications
EN5601

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Lecture 3: Discrete Representation of Continuous Signal, (Part I)

References

- the discussion is essentially based on [§ 2, Benedetto-2008]

Why?

- a key to analysing communication systems
- find a way to replace all waveforms by finite dimensional vectors
- the underlying uncertainties of waveforms can equivalently be analyzed by using those vectors

Orthogonal Expansion of Finite Energy Signals

Insufficiency of Vector Spaces

- vector spaces, the usual notion of basis (i.e., an algebraic basis), and inner product spaces are insufficient [1]
- e.g.,
  - $\mathcal{E} = \mathbb{R}^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers
  - the space of sequences of real numbers
  - can you guess a basis?
- difficulties pile up when we try to extend the notion of basis to an infinite family of vectors

[1] See § 9 of *Mathematics for Physics and Physicists*, Princeton University Press, 2007.

Hilbert Spaces

- the spaces we are going to consider will be of infinite dimension in general
- how to handle those in a well-behaved setting?
- Hilbert and Schmidt $\rightarrow$ the notion of a Hilbert spaces

Hermitian Scalar Product for $\mathcal{L}_2$ Signals

- define the hermitian scalar product of $\mathcal{L}_2$ signals $x(t)$, $y(t)$:

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt \tag{1}$$

- note that $\langle x, y \rangle$ is always finite [2]

- the norm is defined as

$$\sqrt{\langle x, x \rangle} = ||x|| = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} \tag{2}$$

[2] See Exercise 5.8 of [Gallager-2008]

Hermitian Scalar Product for $\mathcal{L}_2$ Signals

- we also have

$$\langle \hat{x}, \hat{y} \rangle = \int_{-\infty}^{\infty} \hat{x}(f)\hat{y}^*(f)df, \tag{3}$$

  where $\hat{x}$ and $\hat{y}$ are the Fourier transforms of x and y

- Parseval's $\implies \langle x, y \rangle = \langle \hat{x}, \hat{y} \rangle$

The $\mathcal{L}_2$ Inner Product Space

- the set of $\mathcal{L}_2$ signals, together with the hermitian scalar product (1) and the norm (2) constitute the $\mathcal{L}_2$ inner product space [3]
- in fact, the $\mathcal{L}_2$ inner product space is a Hilbert space [4]

[3] The vectors in $\mathcal{L}_2$ inner product space are equivalent classes in the set of $\mathcal{L}_2$ functions.

[4] See § 9.3.c and 9.3.d of *Mathematics for Physics and Physicists*, Princeton University Press, 2007.

Orthogonality of $\mathcal{L}_2$ Signals

- signals $x(t)$ and $y(t)$ are orthogonal when $\langle x, y \rangle = 0$
- $\langle x, x \rangle = \mathcal{E}_x =$ the energy of $x(t)$
- if $x(t)$ and $y(t)$ are orthogonal, sum of energies of $x(t)$ and $y(t)$ is simply

$$\langle x + y, x + y \rangle = \mathcal{E}_x + \mathcal{E}_y$$

Sequence of Orthonormal Signals in $\mathcal{L}_2$

- let $\{\psi_n(t)\}_{n\in\mathcal{I}}$ be a sequence of orthogonal $\mathcal{L}_2$ signals such that

$$\langle \psi_n, \psi_m \rangle = \begin{cases} 1, & n = m \\ 0, & \text{otherwise,} \end{cases}$$

  where $\mathcal{I}$ is a countable index set

- signals of this sequence are caller orthonormal

Approximate a Given $\mathcal{L}_2$ Signal

- let $x(t) \in \mathcal{L}_2$ and $\{\psi_n(t)\}_{n\in\mathcal{I}}$ is a orthonormal sequence of $\mathcal{L}_2$ signals
- 'approximate' $x(t)$ by $\bar{x}(t)$, where

$$\bar{x} = \sum_{n\in\mathcal{I}} c_n \psi_n(t)$$

Approximate a Given $\mathcal{L}_2$ Signal

- 'approximate' in what sense?

Approximate a Given $\mathcal{L}_2$ Signal

- 'approximate' in what sense? find $c = (c_n)_{n\in\mathcal{I}}$ to minimize the energy of the error signal $e(t)$, where

$$e(t) = x(t) - \bar{x}(t) = x(t) - \sum_{n\in\mathcal{I}} c_n \psi_n(t)$$

- energy $\mathcal{E}_e(c)$ of the above error signal is

$$\begin{aligned}
\mathcal{E}_e(c) = \langle e, e\rangle &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{n\in\mathcal{I}} c_n \psi_n(t) \right|^2 dt \\
&= \mathcal{E}_x + \mathcal{E}_{\bar{x}} - 2\mathrm{Re}\{\langle x, \bar{x}\rangle\} \\
&= \mathcal{E}_x + \sum_{n\in\mathcal{I}} |c_n|^2 - 2\mathrm{Re}\left\{\sum_{n\in\mathcal{I}} c_i \langle x, \psi_n\rangle\right\} \\
&= \mathcal{E}_x + \sum_{n\in\mathcal{I}} |c_n - \langle x, \psi_n\rangle|^2 - \sum_{n\in\mathcal{I}} |\langle x, \psi_n\rangle|^2
\end{aligned}$$

Approximate a Given $\mathcal{L}_2$ Signal

- thus, $\mathcal{E}_e(c)$ is minimized when $c_n = \langle x, \psi_n \rangle$ for all n
- in this case, the minimum energy $\mathcal{E}_e^\star = \mathcal{E}_x - \sum_{n \in \mathcal{I}} |c_n|^2$
- in this case, we have

$$\bar{x}(t) = \sum_{n \in \mathcal{I}} \langle x, \psi_n \rangle \psi_n(t) \tag{4}$$

Approximate a Given $\mathcal{L}_2$ Signal

- $\bar{x}(t)$ given in (4) is called the projection of $x(t)$ onto the space spanned by the signals $\{\psi_n(t)\}_{n\in\mathcal{I}}$
- this terminology stems from that the corresponding error signal is
  - orthogonal to $\psi_n(t)$ for all $n \in \mathcal{I}$
  - and hence it also orthogonal to $\bar{x}(t)$ in (4)
  - verify ..

A Generalization for Projection

- Theorem [Theorem 5.3.2, Gallager-2008]:

- Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of orthonormal vectors in $\mathcal{L}_2$, and let x be an arbitrary $\mathcal{L}_2$ vector. Then there exists a unique $\mathcal{L}_2$ vector $\bar{x}$ such that $x-\bar{x}$ is orthogonal to each ψ_i and

$$\lim_{m\to\infty}\left|\left|\bar{x}-\sum_{n=1}^{m}c_n\psi_n\right|\right|=0, \quad \text{where } c_n=\langle x,\psi_n\rangle; \tag{5}$$

$$||\bar{x}||^2=\sum|c_n|^2. \tag{6}$$

Conversely, for any complex sequence $\{c_n\}_{n\in\mathbb{N}}$ such that $\sum_n|c_n|^2<\infty$, an $\mathcal{L}_2$ vector $\bar{x}$ exists satisfying (5) and (6). Remark

A Generalization for Projection

- can $\mathcal{E}_e^\star = 0$ hold?

[5] See Theorem 9.21 of *Mathematics for Physics and Physicists*, Princeton University Press, 2007 for details.

A Generalization for Projection

- can $\mathcal{E}_e^\star = 0$ hold?
- not in general
- this holds when the sequence $\{\psi_n(t)\}_{n\in\mathcal{I}}$ is a Hilbert basis [5]

[5] See Theorem 9.21 of *Mathematics for Physics and Physicists*, Princeton University Press, 2007 for details.

Examples

- for time-limited $\mathcal{L}_2$ functions
  - $\psi_n(t) = (1/\sqrt{T})e^{2\pi int/T}$
- in this case the equality

$$x(t) = \text{l.i.m} \sum_n \langle x, \psi_n \rangle \psi_n(t)$$

holds

Examples

- for $\mathcal{L}_2$ functions
  - $\psi_{km}(t) = (1/\sqrt{T})e^{2\pi ikt/T}\,\mathsf{rect}\left(\frac{t}{T} - m\right)$
- in this case the equality

$$x(t) = \text{l.i.m} \sum_k \sum_m \langle x, \psi_{km} \rangle \psi_{km}(t)$$

  holds
- called the T-spaced truncated sinusoid expansion
- read pages 112-113 of [Gallager-2008]

Examples

- for $\mathcal{L}_2$ functions
  - $\psi_{km}(t) = (1/\sqrt{T})e^{2\pi imt/T}\text{sinc}\left(\frac{t}{T} - k\right)$
- in this case the equality

$$x(t) = \text{l.i.m} \sum_k \sum_m \langle x, \psi_{km} \rangle \psi_{km}(t) \tag{7}$$

  holds

- called the T-spaced sinc-weighted sinusoid expansion
- read pages 126-127 of [Gallager-2008]

Expansion of Bandlimited Signals

- consider an $\mathcal{L}_2$ signal which is strictly bandlimited in the interval $[-W, W]$
- if we set $W = 1/2T$, this is a particular case of T-spaced sinc-weighted sinusoid expansion

Expansion of Bandlimited Signals

- in particular, the second summation of (7) is no more present and we have

$$x(t) = \text{l.i.m} \sum_k \langle x, \psi_{k0} \rangle \psi_{k0}(t)$$
$$= \text{l.i.m} \sum_k x(kT) \text{sinc}\left(\frac{t}{T} - k\right) \quad (8)$$

  holds

- (8) is simply the sampling expansion of bandlimited signals

- read pages 76-77 of [Benedetto-2008] and compare the content with those in pages 126-127 of [Gallager-2008]

Gram-Schmidt Procedure

- an algorithm for constructing orthogonal sequences
- called Gram-Schmidt orthogonalization procedure
- let $\{\phi_n(t)\}_{n=1}^N$ be a sequence of finite energy signals
- assume the signals are linearly independent [6]
- then Gram-Schmidt Procedure can be used to construct an orthonormal sequence of finite energy signals $\{\psi_n(t)\}_{n=1}^N$ that spans the subspace spanned by $\{\phi_n(t)\}_{n=1}^N$
- read page 73 of [Benedetto-2008]

[6] i.e., $||\sum_n c_n\phi_n(t)|| = 0 \implies \forall n\ c_n = 0.$

Orthogonal Expansion of Random Signals (next lecture)