Digital Communications
EN5601

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Lecture 2: Channel Waveforms

Why?

- to begin studying the waveforms that are typically transmitted at the input and received at the output of communication channels
  - channels: black boxes where the output waveform can be described as a function of the input waveform and noise
- also important to understand analog data compression (out of the scope of this module)

Fourier Series

Fourier Series

- the Fourier series of time-limited function maps function to a sequence of coefficients
- let $u(t) = 0$ for $T/2 < t < -T/2$, then

$$u(t) \underset{?}{=} \sum_{k=-\infty}^{\infty} \hat{u}_k e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T}\right) \tag{1}$$

- the complex coefficient $\hat{u}_k$ is given by

$$\hat{u}_k \underset{?}{=} \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi i k t/T} dt, \quad -\infty < k < \infty$$

Fourier Series

- this is not necessarily for periodic signals, though many of us taught like that
- more importantly, take a time-limited function and map it into a sequence of coefficients
- i.e., converting a time-limited function to a sequence of bits, e.g., speech coding

Fourier Series

- what functions can be represented as (1), and what can not ?
- example rect$(2t)$
- $u(t)$ is its Fourier series except at $t = \pm 1/4$
- when some discontinuities there in $u(t)$, then some problems

Fourier Series

- in fact

$$u(t) \neq \sum_{k=-\infty}^{\infty} \hat{u}_k e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T}\right)$$

- however, in engineering applications, this is not a problem

- such peculiarities can gracefully be handled by the theories of $\mathcal{L}_2$ spaces

Fourier Series

- in particular, two functions are equivalent in $\mathcal{L}_2$ sense if their energy difference has zero energy, i.e.,

$$\int_{-\infty}^{\infty} |u(t) - v(t)|^2 dt = 0$$

- thus,

$$u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T}\right)$$

  in $\mathcal{L}_2$ sense

- thus, there are functions with same Fourier series coefficients, and they are $\mathcal{L}_2$ equivalent

Fourier Series

- not all time-limited functions have Fourier series, even in the sense of $\mathcal{L}_2$ equivalence
- in general when does a given time-limited function have Fourier series?
- an important class is finite energy functions
- i.e., functions that are square integrable

Fourier Series

- Theorem [Theorem 4.4.1, Gallager-2008]:
  - Let $u : [-T/2, T/2] \to \mathbb{C}$ be an $\mathcal{L}_2$ function. Then for each $k \in \mathbb{Z}$, the Lebesgue [1] integral

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi i k t/T} dt \tag{2}$$

exists as a finite complex number. Furthermore,

$$\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \textstyle\sum_{k=-k_0}^{k_0} \hat{u}_k e^{2\pi i k t/T} \right|^2 dt = 0. \tag{3}$$

Also the energy equation $\int_{-T/2}^{T/2} |u(t)|^2 = T \sum_{k=-\infty}^{\infty} |\hat{u}_k|^2$ holds. Finally, if $\{\hat{u}_k\}_{k \in \mathbb{Z}}$ is a sequence of complex numbers satisfying $\sum_{k=-\infty}^{\infty} |\hat{u}_k|^2 < \infty$, then an $\mathcal{L}_2$ function $u : [-T/2, T/2] \to \mathbb{C}$ exists satisfying the above.

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[1] See § 4.3, Gallager-2008 for details. The § 2 of *Mathematics for Physics and Physicists*, Princeton University Press, 2007 has a comprehensive dissemination of the topic.

FS Coefficients and Convergence

- note that the Fourier coefficient $\hat{u}_k$ given in (2) must exist and both real and imaginary parts are finite for all $k \in \mathbb{Z}$ [2]
- discuss the meaning of (3)
- the convergence of the form of (3) is referred to as convergence in the mean, and is denoted by l.i.m

[2] See pages 26-30 for some details.

FS Coefficients and Convergence

- thus Fourier Series is written as

$$u(t) = \text{l.i.m} \sum_{k=-\infty}^{\infty} \hat{u}_k e^{2\pi ikt/T} \text{rect}\left(\frac{t}{T}\right)$$

where

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2pikt/T} dt, \quad -\infty < k < \infty$$

- one can segment an arbitrary $\mathcal{L}_2$ function into segments of width T

- each segment can be represented by a finite number [3] of FS coefficients

[3] The number of coefficients may determined by the application requirement, e.g., acceptable MSE.

An Orthogonal Expansion for $\mathcal{L}_2$ Functions

- consider the segmentation pointed in the previous slide
- thus, any arbitrary $\mathcal{L}_2$ function $u(t)$ is given by

$$u(t) = \text{l.i.m} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{u}_{km} e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T} - m\right)$$

- a conceptual basis for voice compression algorithms

Fourier Transform

Fourier Transform

- let $u(t) : \mathbb{R} \to \mathbb{C}$, $\hat{u}(f) : \mathbb{R} \to \mathbb{C}$

$$\hat{u}(f) \underset{?}{=} \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} dt \tag{4}$$

$$u(t) \underset{?}{=} \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t} df \tag{5}$$

- for well-behaved functions both integrals above exist

Special Cases

- $u(0)$ and $\hat{u}(0)$ are given by

$$u(0) = \int_{-\infty}^{\infty} \hat{u}(f)df$$

$$\hat{u}(0) = \int_{-\infty}^{\infty} u(t)dt$$

- Parseval's theorem:

$$\int_{-\infty}^{\infty} u(t)v^*(t)dt = \int_{-\infty}^{\infty} \hat{u}(f)\hat{v}^*(f)df$$

- energy equations:

$$\int_{-\infty}^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2 df$$

Fourier Transform, Existence

- in particular,
  - $u(t)$ is $\mathcal{L}_1 \implies$
    - (4) exists for all $f \in \mathbb{R}$
    - $\hat{u}(f)$ is continuous [4]
  - $\hat{u}(f)$ is $\mathcal{L}_1 \implies$
    - (5) exists for all $t \in \mathbb{R}$
    - $u(t)$ is continuous

[4] See Theorem 3.11 of *Mathematics for Physics and Physicists*, Princeton University Press, 2007.

Fourier Transform

- in general, FT of an $\mathcal{L}_1$ function is not necessarily $\mathcal{L}_1$, e.g., rect(t)
- $\mathcal{L}_1$ is not enough to when modelling communication systems, e.g., sinc is not $\mathcal{L}_1$
- functions with discontinuities cannot be FT of any $\mathcal{L}_1$ function
- $\mathcal{L}_1$ functions might have infinite energy

$\mathcal{L}_2$ Functions & Fourier Transform

- $\mathcal{L}_2$ turns out to be the right choice
- the Fourier transform (4) does not necessarily exist at each f
- but it does exist as an $\mathcal{L}_2$ limit
- the inverse transform (5) does exist in exactly the same sense

$\mathcal{L}_2$ Functions & Fourier Transform

- consider any $\mathcal{L}_2$ function $u : \mathbb{R} \to \mathbb{C}$ and define a related function u_a as follows:

$$u_a(t) = u(t)\text{rect}\left(\frac{t}{2a}\right), \ a > 0$$

- u_a is $\mathcal{L}_2$ and time-limited $\implies$ u_a is $\mathcal{L}_1$

- as a result, the FT $\hat{u}_a(f)$ of $u_a(t)$ exists and continuous, where,

$$\hat{u}_a(f) = \int_{-\infty}^{\infty} u_a(t)e^{-2\pi ift}dt = \int_{-a}^{a} u(t)e^{-2\pi ift}dt$$

- now the idea is to let $a \to \infty$ in above equation to get the Fourier transform $\hat{u}(f)$

$\mathcal{L}_2$ Functions & Fourier Transform

- recall... the idea is to let $a \to \infty$ in above equation to get the Fourier transform $\hat{u}(f)$
- but this limit $\hat{u}(f)$ does not necessarily exist for each f
- Plancherel's theorem, however, asserts that this limit exists in $\mathcal{L}_2$ sense

$\mathcal{L}_2$ Functions & Fourier Transform

- Theorem [Theorem 4.5.1, Gallager-2008]:
  - There is an $\mathcal{L}_2$ function $\hat{u}(f)$ (the FT of $u(t)$), which satisfies

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} |\hat{u}(f) - \hat{u}_a(f)|^2 \, df = 0.$$

    Also the energy equation $\int_{-\infty}^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2 df$ holds.

- guarantees the existence of a FT (up to $\mathcal{L}_2$-equivalence)
- guarantees that it is arbitrarily closely approximated (in difference energy) by the continuous Fourier transforms of the truncated versions of $u(t)$

$\mathcal{L}_2$ Functions & Fourier Transform

- Theorem [Theorem 4.5.2, Gallager-2008]:

- For an $\mathcal{L}_2$ function $u : \mathbb{R} \to \mathbb{C}$, let $\hat{u}(f)$ be the FT given as in the above theorem. Let $u_b : \mathbb{R} \to \mathbb{C}$ defined as

$$u_b(t) = \int_{-b}^{b} \hat{u}(f) e^{2\pi i f t} df.$$

Then

$$\lim_{b \to \infty} \int_{-\infty}^{\infty} |u(t) - u_b(t)|^2 \, dt = 0.$$

- guarantees the existence of a inverse FT (up to $\mathcal{L}_2$-equivalence)

- guarantees that it is arbitrarily closely approximated (in difference energy) by the continuous inverse Fourier transforms of the truncated versions of $\hat{u}(f)$

$\mathcal{L}_2$ Functions & Fourier Transform

- both parts of the Plancherel theorem given above state that every $\mathcal{L}_2$ function $u(t)$ has a Fourier transform $\hat{u}(f)$ (an $\mathcal{L}_2$ function again) satisfying

$$\hat{u}(f) = \text{l.i.m} \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} dt$$

$$u(t) = \text{l.i.m} \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t} df$$

Appendix [5]

[5] The content in this section is just to give some pointers to the related terminologies and theories. For a pedagogical treatment, we refer the reader to § 2 of *Mathematics for Physics and Physicists*, Princeton University Press, 2007.

Riemann's Vs Lebesgue Integral

- whenever Riemann's integral exists the Lebesgue integral also exists and has the same value
- familiar rules for calculating Riemann's integral also apply for Lebesgue integral
- for some peculiar functions, the Lebesgue integral exists, but Riemann's does not
- to justify the swapping sum and integral, one needs very stringent condition
- for some weird cases not even Lebesgue integral exists

Lebesgue Measure

- for any real $a, b \in \bar{\mathbb{R}}$ [6] with $a \leq b$, the interval (a, b) has a measure $b - a$
- we have the same measure for $(a, b]$, $[a, b)$, and $[a, b]$
- measure of a countable union of disjoint intervals is thesum of the measure of each interval
- the measure of a set $\mathcal{S} \subset \bar{\mathbb{R}}$ is zero if for all $\delta > 0$, $\mathcal{S}$ is contained in a countable union of disjoint intervals $\mathcal{U}$ with measure of $\mathcal{U}$, denoted $\mu(\mathcal{U}) < \delta$

[6] $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

Measurable Functions

- a function $u : \mathbb{R} \to \mathbb{R}$ is measurable if $\{t | a \leq u(t) \leq b\}$ is measurable for each $a, b \in \mathbb{R}$
- suppose $u : \mathbb{R} \to \mathbb{R}^+ \cup \infty$ is measurable
  - then the function is integrable if the 'limit' in the following figure exists and is finite (draw the figure)
- integrability of functions $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ are also defined in a similar manner

Some Properties and Definitions

- consider a function $u : \mathbb{R} \to \mathbb{R}$
- then the functions defined by $|u(t)|$ and $|u(t)|^2$ are nonnegative
- the functions $|u(t)|$ and $|u(t)|^2$ are measurable if $u(t)$ is
- we say
  - $u(t)$ is $\mathcal{L}_1$ if it is measurable and $\int |u(t)|dt < \infty$
  - $u(t)$ is $\mathcal{L}_2$ if it is measurable and $\int |u(t)|^2 dt < \infty$

Some Properties and Definitions

- a complex function $u : \mathbb{R} \to \mathbb{C}$ is measurable if both Re$\{u(t)\}$ Im$\{u(t)\}$ are measurable
- we say
  - $u(t)$ is $\mathcal{L}_1$ if it is measurable and $\int |u(t)|dt < \infty$
  - $u(t)$ is $\mathcal{L}_2$ if it is measurable and $\int |u(t)|^2 dt < \infty$

Some Relationship Between $\mathcal{L}_1$ and $\mathcal{L}_2$

- $|u(t)| \geq 1 \implies |u(t)| \leq |u(t)|^2 \implies |u(t)| \leq |u(t)|^2 + 1$
- $|u(t)| < 1 \implies |u(t)| \leq |u(t)|^2 + 1$
- in other words, $|u(t)| \leq |u(t)|^2 + 1$ for all t
- suppose $u(t)$ is time-limited to $[-T/2, T/2]$, thus we have

$$\begin{aligned}\int_{-T/2}^{T/2} |u(t)| dt &\leq \int_{-T/2}^{T/2} [|u(t)^2 + 1] dt \\ &= T + \int_{-T/2}^{T/2} |u(t)|^2 dt\end{aligned}$$

- therefore, $\mathcal{L}_2$ time-limited functions are also $\mathcal{L}_1$

Existence of Some Integrals

- note that $|u(t)| = |u(t)e^{2\pi i f t}|$
- suppose $u : \mathbb{R} \to \mathbb{C}$ is $\mathcal{L}_1$
  - argue why

$$\int \left|u(t)e^{2\pi i f t}\right| dt < \infty$$

  - argue why

$$\int \left|\text{Re}\{u(t)e^{2\pi i f t}\}\right| dt < \infty \quad \text{and} \quad \int \left|\text{Im}\{u(t)e^{2\pi i f t}\}\right| dt < \infty$$

Existence of Some Integrals

- recall.. $u : \mathbb{R} \to \mathbb{C}$ is $\mathcal{L}_1$
  - argue why

$$\int \text{Re}\{u(t)e^{2\pi ift}\}dt \quad \text{and} \quad \int \text{Im}\{u(t)e^{2\pi ift}\}dt$$

    are finite
  - argue why the the integral

$$\int u(t)e^{2\pi ift}dt$$

    is finite
- suppose $u(t)$ is $\mathcal{L}_2$ and time-limited, then it is $\mathcal{L}_1$, i.e., all the above properties hold