Digital Communications
EN5601

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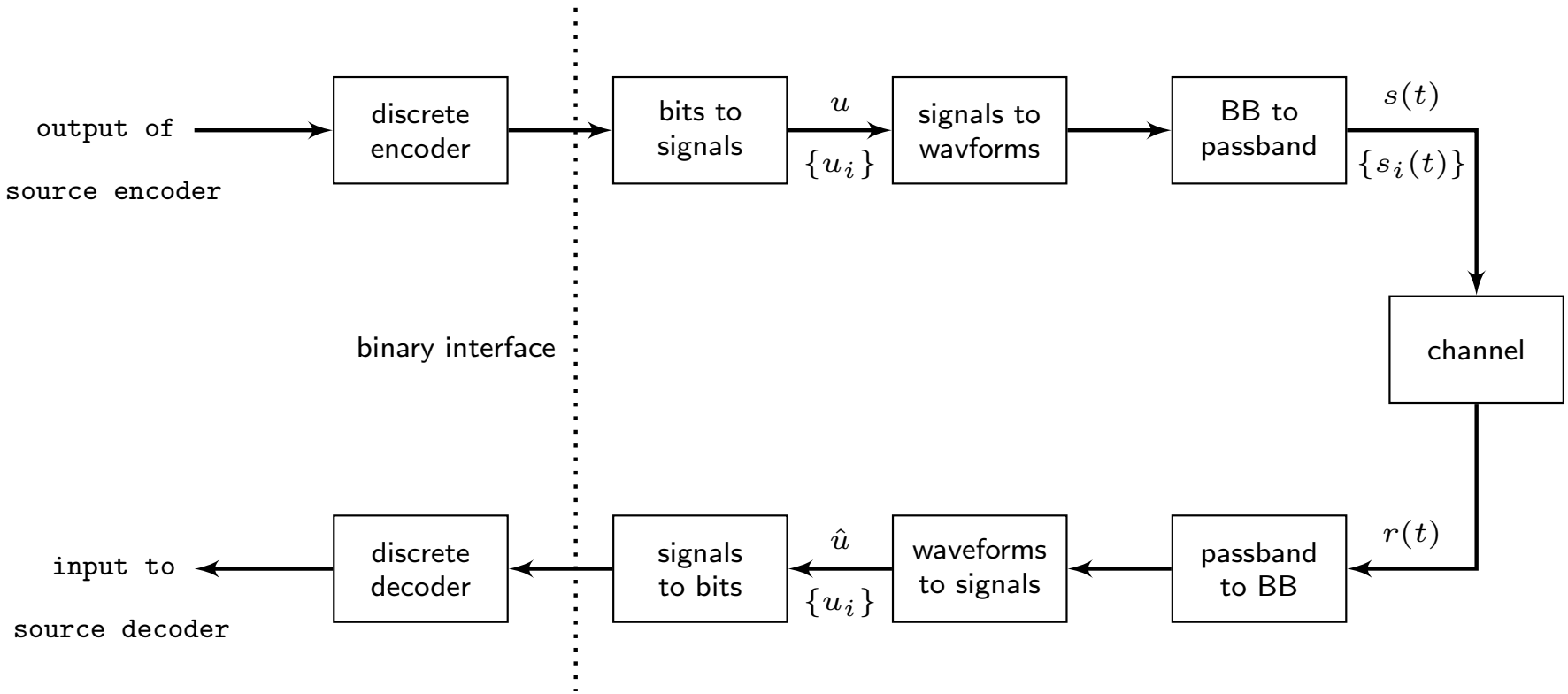
June 2020

Lecture 11: Signals in White Gaussian Noise

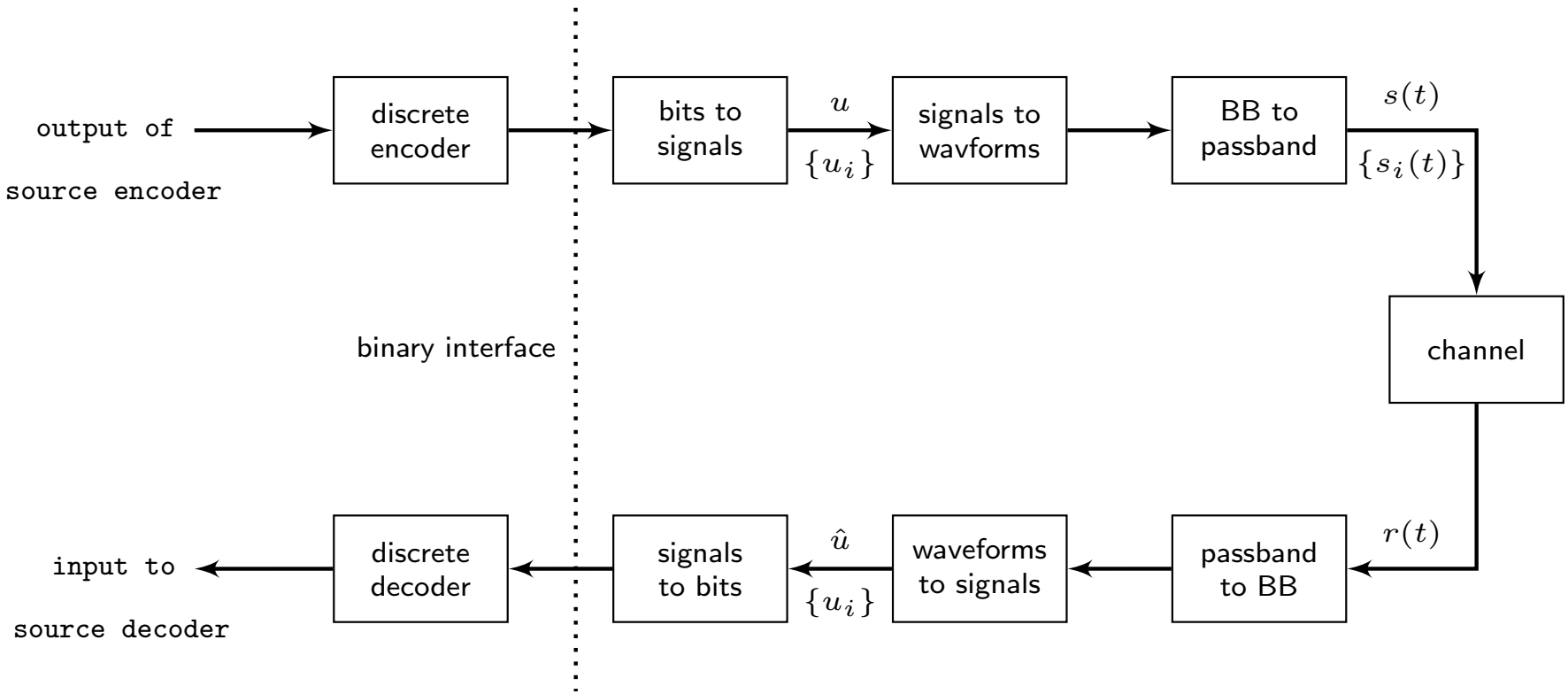
References

- the discussion is essentially based on [§ 4, Wozencraft-1965], [§ 4.3, Bigliery-2008], [§ 5 Wozencraft-1965]

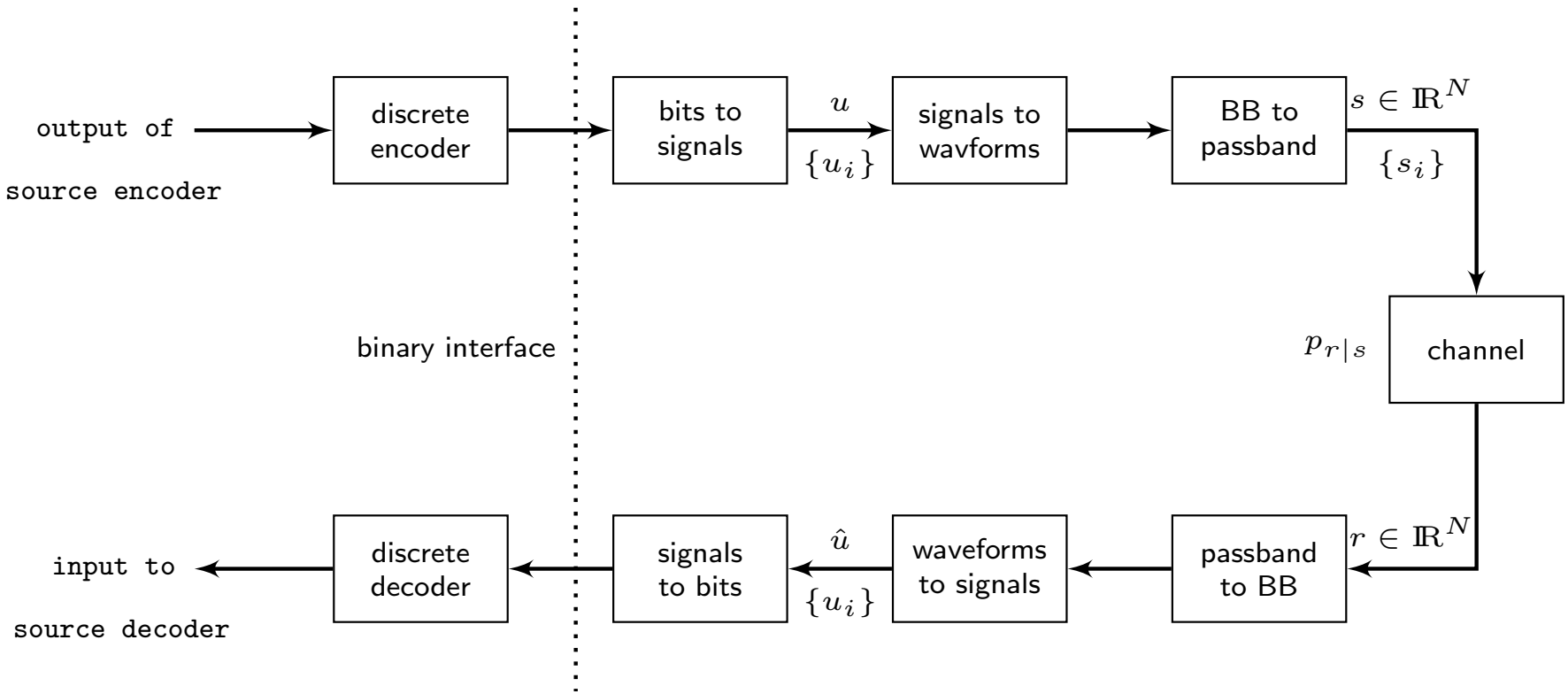
Channel Encoder: A Layering



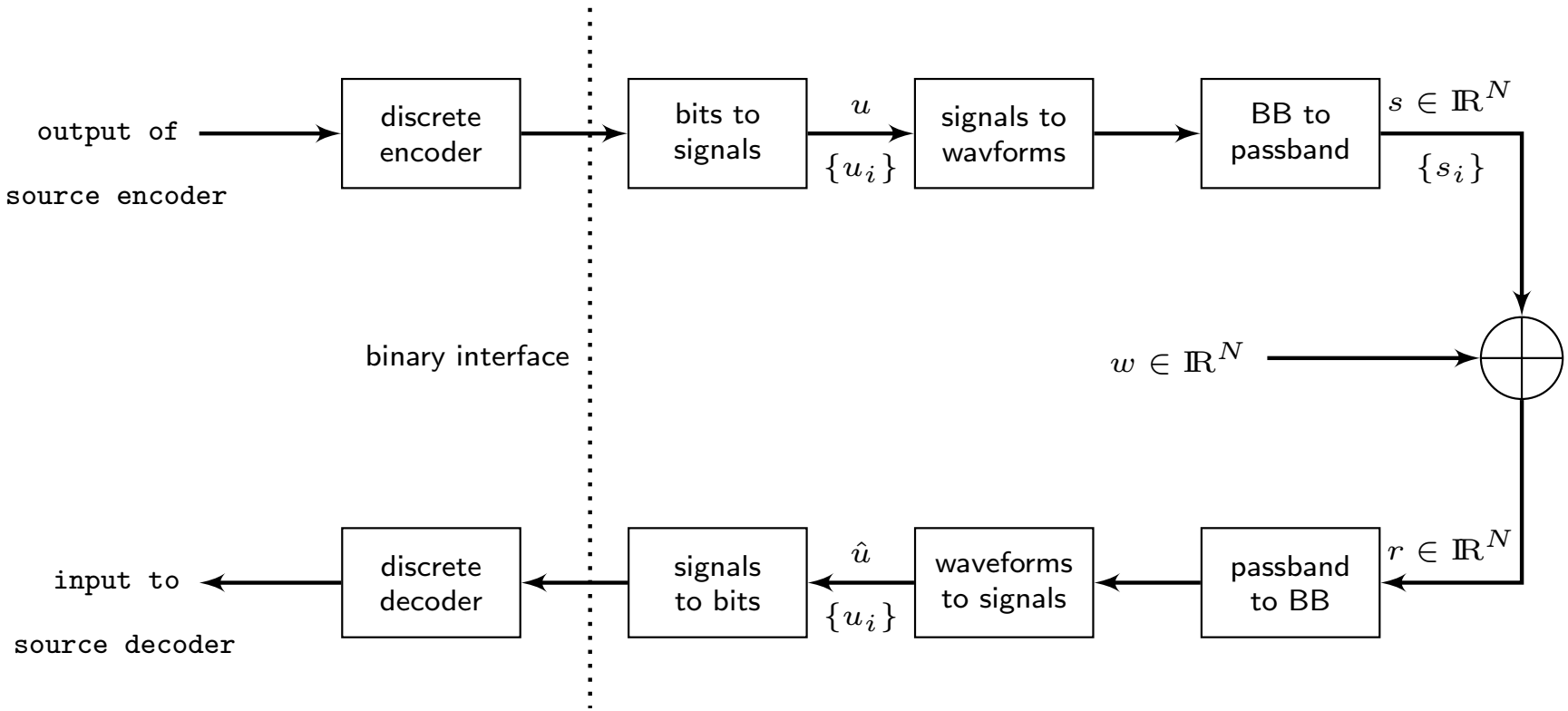
Waveform Channel



Vector Channel



Vector AWGN Channel



Vector AWGN Channel

- noise vector is $w \in \mathbb{R}^N$ is given by

$$w = (w_1, w_2, \ldots, w_N)$$

- the receive vector is $r \in \mathbb{R}^N$ and signal vector $s \in \mathbb{R}^N$ are related by

$$r = s + w$$

Compute Conditional Probability Density Functions

Compute $p_{r|s}$

- conditional probability density functions $p_{r|s=s_i}$, $i = 0, 1, \ldots, M-1$ are to be computed
- the simple relation $r = s + w$, together with that w a random vector with pdf p_w entails that

$$\begin{aligned} p_r(r \mid s = s_i) &= p_w(r - s_i \mid s = s_i) \\ &= p_w(r - s_i) \end{aligned}$$

- the last equality follows from that s and w are independent

Compute $p_{r|s}$

- what is p_w, the noise density ?
- the coefficients of a WGN process in any orthonormal expansion are iid Gaussian, see page 41 of lecture slides EN5601-4
- thus N components of w are statistically independent, zero mean, Gaussian rvs, each with variance σ^2, i.e.,

$$p_w(w) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} ||w||^2\right)$$

Compute $p_{r|s}$

- therefore, we have

$$p_r(r \mid s = s_i) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} ||r - s_i||^2\right)$$

MAP for Vector AWGN Channel

MAP Rule

- recall: on observing $r = \rho$, set $\hat{u}(\rho) = u_k$ whenever

$$P[u_k]\ p_r(\rho \mid s = s_k) \geq P[u_i]\ p_r(\rho \mid s = s_i) \qquad \forall i \neq k$$

- the above rule is equivalent to: set $\hat{u}(\rho) = u_k$ whenever

$$||\rho - s_k||^2 - 2\sigma^2\ \ln P[u_k] \leq ||\rho - s_i||^2 - 2\sigma^2\ \ln P[u_i] \qquad \forall i \neq k$$

- in the case of equi-probable symbols, i.e., $P[u_i] = P[u_j]$ for all $i, j \in \{0, 1, \ldots, M-1\}$, the MAP $\rightarrow$ ML

- ML rule: set $\hat{u}(\rho) = u_k$ whenever

$$||\rho - s_k|| \leq ||\rho - s_i|| \qquad \forall i \neq k$$

Decision Regions for Vector AWGN Channel

Decision Regions

- discuss the idea with an example

Probability of Error for Vector AWGN Channel

Probability of Correct Decision

- once the decision regions $\{\mathcal{I}_k\}$ are determined, as expression for the conditional probability of correct decision follows
- we have

$$P[\mathcal{C} \mid u_k] = P[r \in \mathcal{I}_k \mid u_k] = \int_{\mathcal{I}_k} p_r(r \mid s = s_i)\, dr$$

$$= \int_{\mathcal{I}_k} \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} ||r - s_i||^2\right) dr$$

Probability of Error

- thus the probability of error is simply

$$P[\mathcal{E}] = 1 - \sum_{k=0}^{M-1} P[\mathcal{C} \mid u_k] \, P[u_k]$$

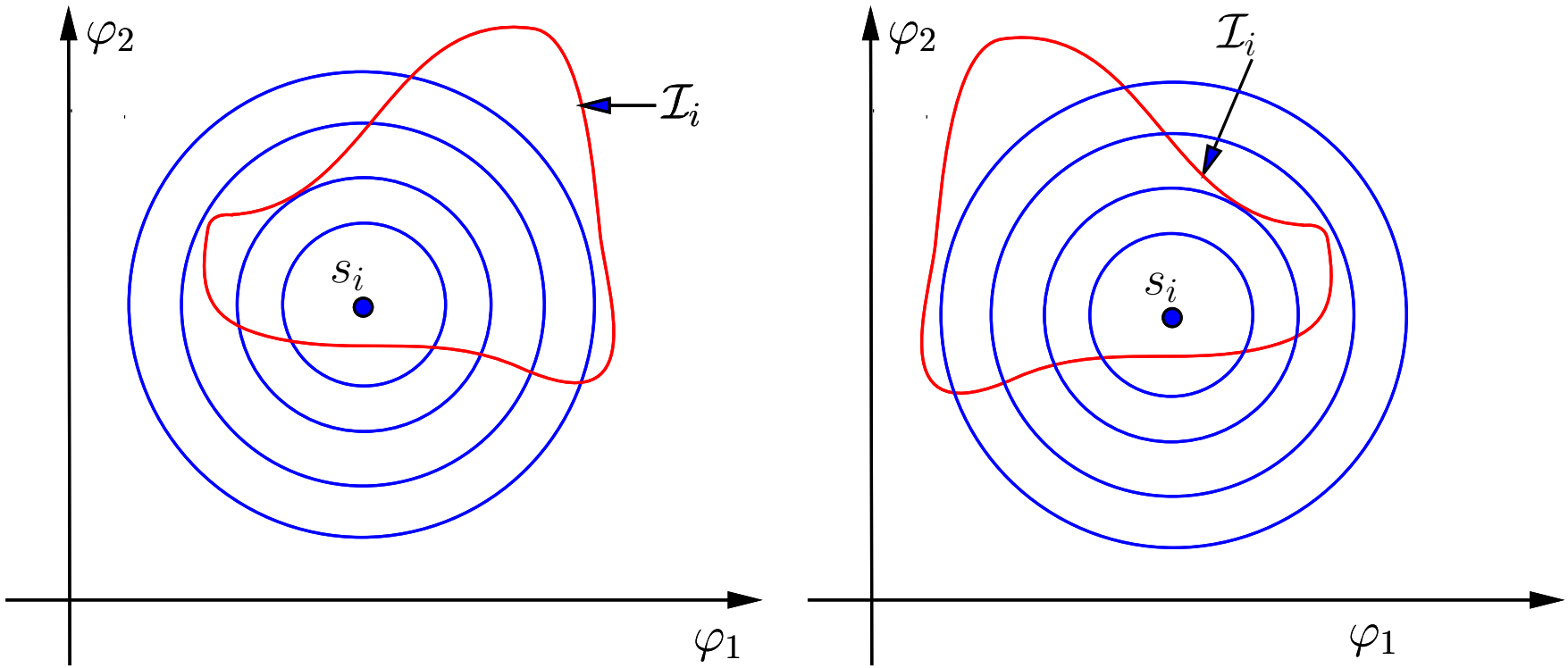
Equivalent Signal Sets

- signal sets can be equivalent by virtue of their geometric configuration
- but different geometrical configurations may also be equivalent in terms of $P[\mathcal{E}]$, e.g.,
  - rotation
  - translation

Rotation and Translation

- the probability of $s_i + w$ falls within $\mathcal{I}_i$ is unaffected if s_i and $\mathcal{I}_i$ are translated together through signal space
  - why? noise is additive and is independent of the signal
- the same probability is unaffected if $\mathcal{I}_i$ rotates about s_i
  - why? noise is spherically symmetric

Rotation and Translation



Minimum Energy Signal Set

- the probability of correct decision is invariant to translation
- but the aforementioned translation affect the energy required to transmit each signal
  - let $s_i(t)$ denote the passband real signal whose vector representation is s_i
  - the energy E_{s_i} is given by

$$E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t)\, dt = ||s_i||^2 \neq ||s_i - a||^2,$$

where a is a nonzero translation on s_i

Minimum Energy Signal Set

- what is optimal a such that the average energy $E_s(a)$

$$E_s(a) = \sum_{k=0}^{M-1} P[u_k] \; ||s_k - a||^2$$

is minimized ?

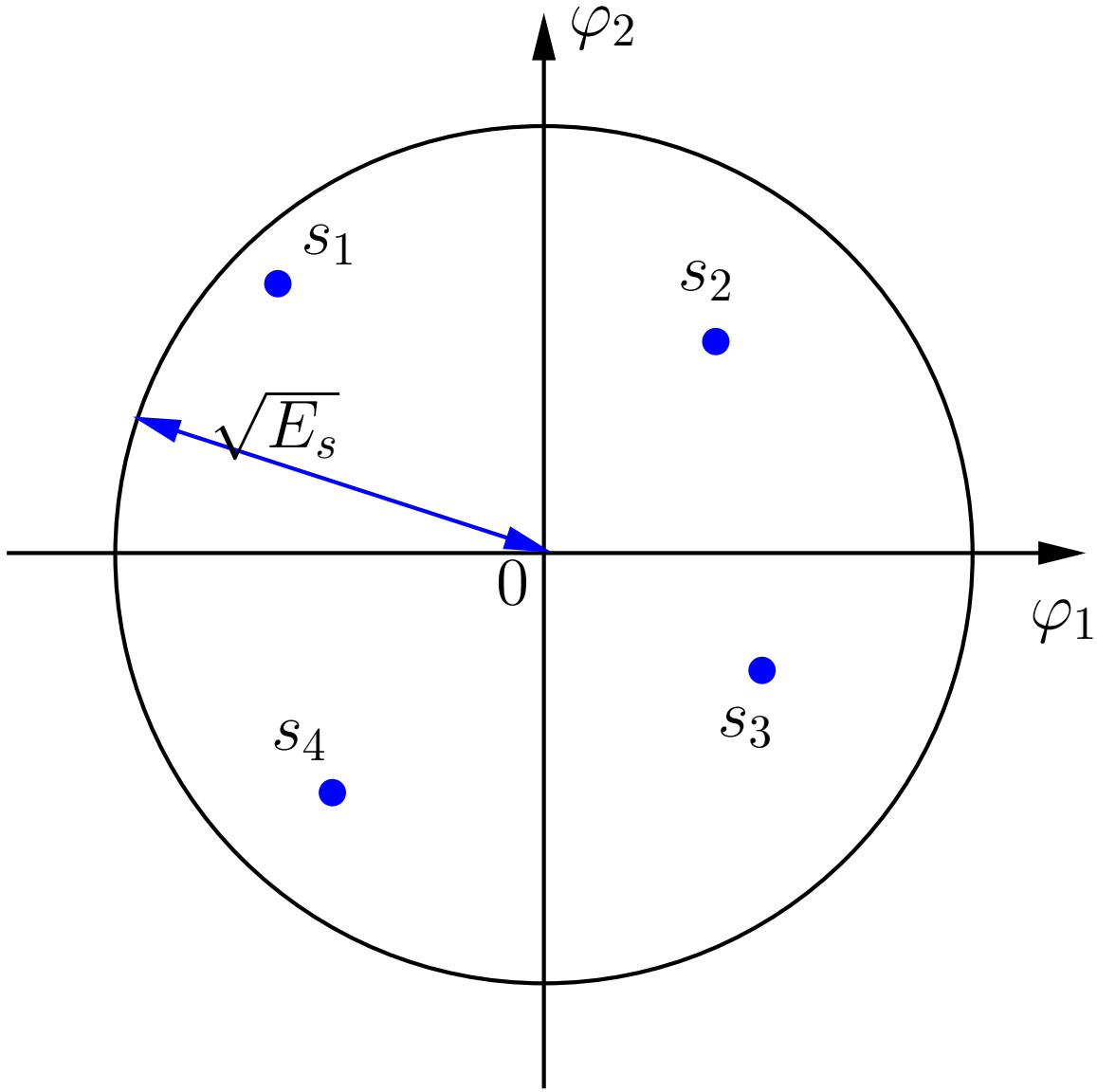
$$\underset{a}{\text{minimize}} \sum_{k=0}^{M-1} P[u_k] \; ||s_k - a||^2$$

- equivalent to letting a be the centroid of the signal set

Peak Signal Energy Constraints

- suppose there is a constrain on the peak energy $E_{\max}$ permitted for any signal point, i.e.,
  - $\max_k \ ||s_k||^2 \leq E_{\max}$

Minimum Energy Signal Set



Probability of Error

- discuss pp. 12-20 of lecture slides EN5601-8