Digital Communications
EN5601

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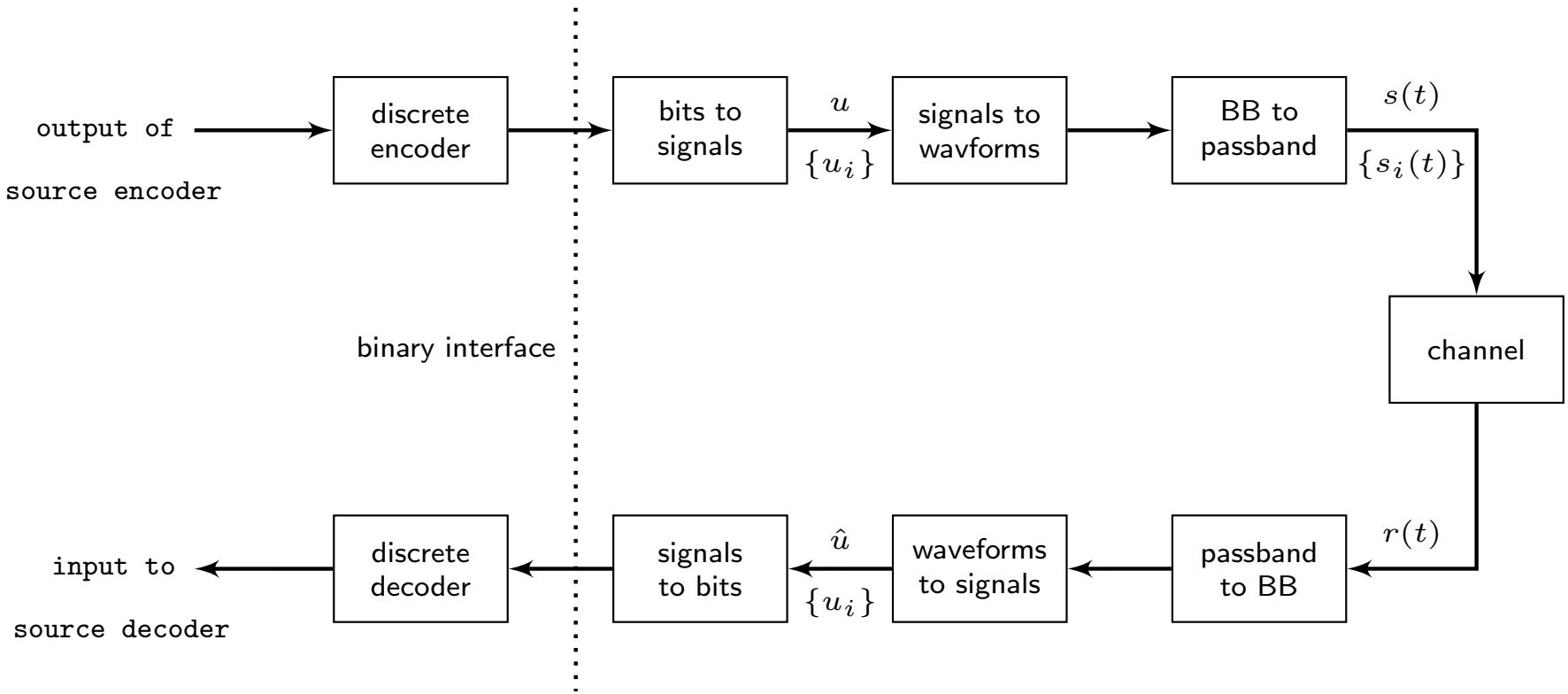
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Lecture 10: Optimum Receiver Principles

References

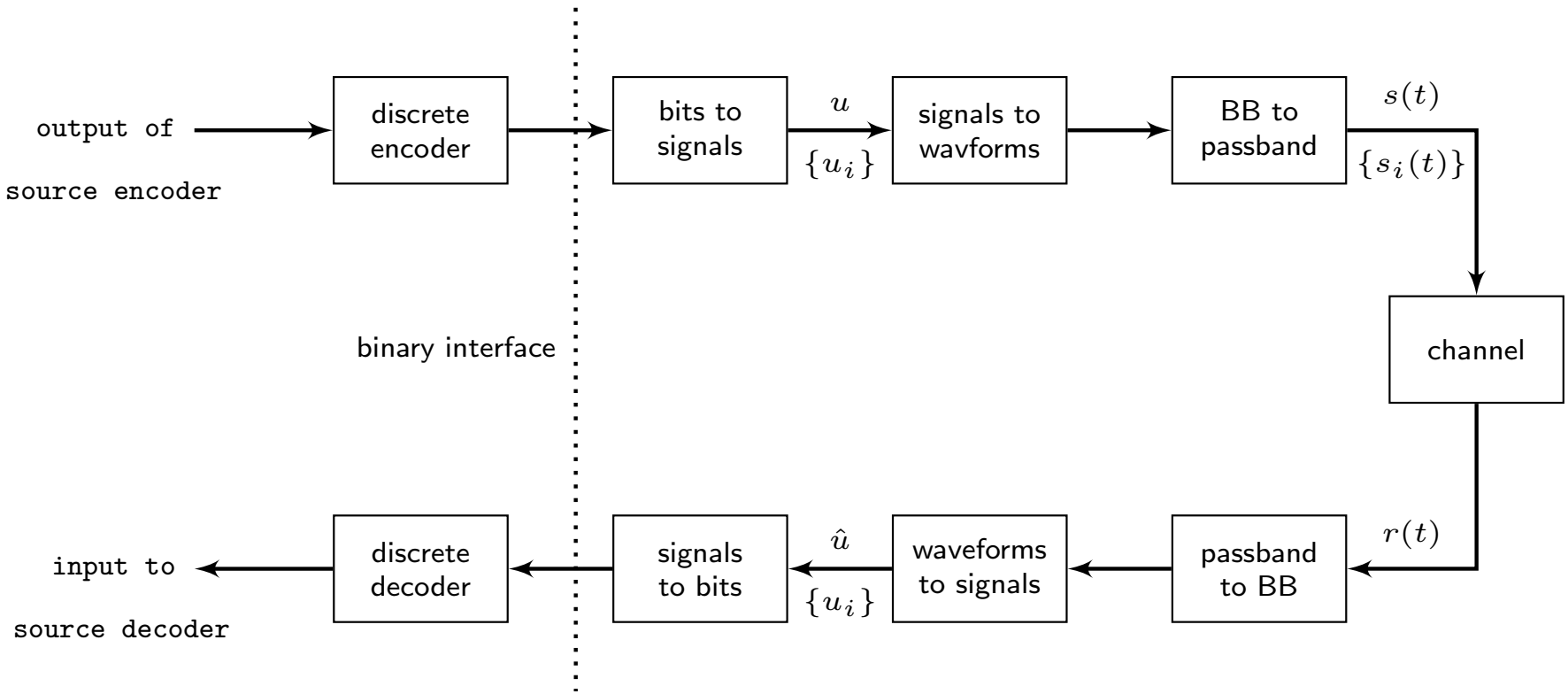
- the discussion is essentially based on [§ 4, Wozencraft-1965], [§ 4.3, Bigliery-2008], [§ 5 Wozencraft-1965]

Channel Encoder: A Layering



Waveform Channel

Waveform Channel



Received Signal

- discrete set of specified waveforms $\{s_i(t)\}_{i=0,\ldots,M-1}$, is transmitted over a channel disturbed by some random phenomena
- e.g., AWGN
  - the received signal in the case of AWGN is given by

$$r(t) = s(t) + n_w(t) \tag{1}$$

  - $n_w(t)$ is a zero-mean Gaussian process with power density

$$S_{n_w}(f) = N_0/2$$

Optimal Receiver

Optimum Receiver

- which waveform is actually transmitted $\rightarrow$ depends on the random signal u
- when $u = u_i$ the transmitted waveform is $s_i(t)$, i.e.,

$$u = u_i \iff s(t) = s_i(t) \tag{2}$$

- the a priori probabilities $\{P[u_i]\}$ specify the input source, where

$$P[u_i] = P[u_i \text{ is transmitted}] \tag{3}$$

Optimum Receiver

- the received signal $r(t)$ should be processed to produce an estimate $\hat{u}$ of the transmitter input u

Optimum Receiver

- the estimate is optimum in the sense that the probability of error

$$P[\mathcal{E}] = P[\hat{u} \neq u] \tag{4}$$

  is minimum

- the goal is to determine the optimum receiver architecture

Optimum Receiver

- to yield the above optimality, what operations are to be performed on $r(t)$?

Basic Approach to Optimal Receiver Design

Basic Approach

- $r(t)$ is a random process
- key to analyse the situation: replace all waveforms by final dimensional vectors
  - as a result, joint densities can be computed
  - system can be modelled by using theories of probability
  - the analysis is more tractable
- is the aforementioned replacing permissible?

[1] This is based on orthonormal expansions and vector representations of signals.

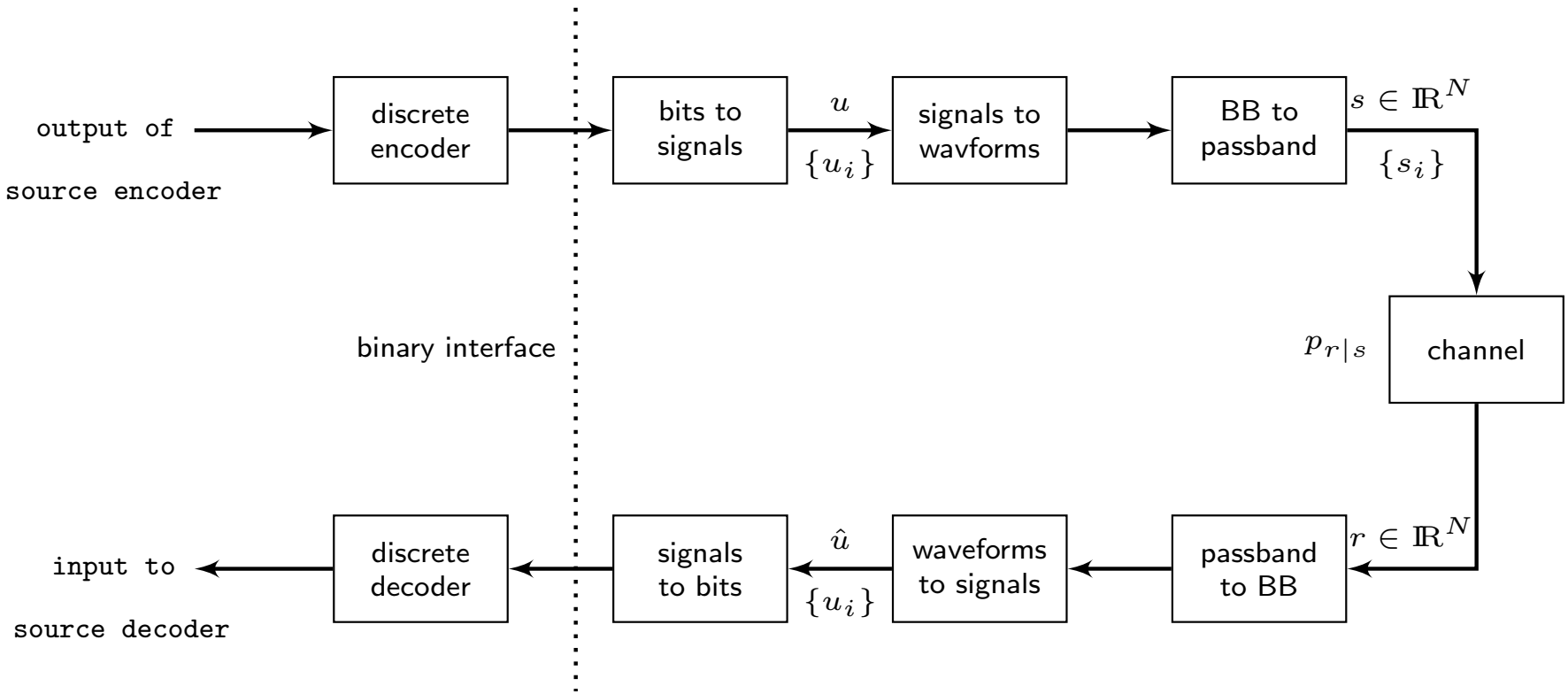
Basic Approach

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- key to analyse the situation: replace all waveforms by final dimensional vectors
  - as a result, joint densities can be computed
  - system can be modelled by using theories of probability
  - the analysis is more tractable
- is the aforementioned replacing permissible? YES [1]

[1] This is based on orthonormal expansions and vector representations of signals.

Vector Channel

Vector Channel



Vector Channel

- M signal vectors, i.e., $s_i \in \mathbb{R}^N$, $i = 0, 1, \ldots, M-1$ where

$$s_i = (s_{i1}, s_{i2}, \ldots, s_{iN})$$

- the vectors are described in terms of coordinates φ_j, $j = 1, \ldots, N$

- when $u = u_i$, the vector s_i is transmitted

- the channel is characterized by the conditional probability density functions $p_{r|s=s_i}$, $i = 0, 1, \ldots, M-1$, where

$$r = (r_{i1}, r_{i2}, \ldots, r_{iN})$$

Examples

- in an AWGN channel
  - what is s and r for M-PAM
  - what is s and r for M-QAM
  - what is s and r for M-FSK

Vector Channel

- assumptions
  - a priori probabilities $\{P[u_i]\}_{i=0,\ldots,M-1}$ are known
  - signals $\{s_i(t)\}_{i=0,\ldots,M-1}$ are known
  - conditional probability density functions $p_{r|s=s_i}$, $i = 0, 1, \ldots, M-1$, are known

MAP Rule for Minimum Probability of Error

Probability of Error

- recall $P[\mathcal{E}]$ is to be minimized, i.e., the optimum receiver
- $P[\mathcal{E}] = 1 - P[\mathcal{C}] \qquad \rightarrow \qquad P[\mathcal{C}]$ is to be maximized
- $\mathcal{C}$ is the event of having the correct decision [2]

[2] Compare with (4).

Probability of Error

- $P[\mathcal{C}]$ is given by

$$P[\mathcal{C}] = \int_{-\infty}^{\infty} P[\ \mathcal{C} \mid r = \rho\ ]\ p_r(\rho)\ d\rho \tag{5}$$

$$= \int_{-\infty}^{\infty} P[\ \hat{u}(\rho) \mid r = \rho\ ]\ p_r(\rho)\ d\rho \tag{6}$$

- here $\hat{u}(\rho)$ is the decision made by the receiver
- the receiver observes ρ (but not what is Txed) and maps ρ into a decision $\hat{u}(\rho) \in \{u_0, u_1, \ldots, u_{M-1}\}$
- note that the decision is correct if $\hat{u}(\rho)$ has been transmitted

Probability of Error

- $P[\ \hat{u}(\rho) \mid r = \rho\ ]$ is the probability that $\hat{u}(\rho)$ is transmitted conditional on the observation $r = \rho$
- note that $p_r(\rho) \geq$ for all ρ
- to maximize (6), we have to maximize

$$P[\hat{u}(\rho) \mid r = \rho] \quad \text{for all } \rho$$

Probability of Error

- recall from assumption on page 14, it is possible to compute the a posteriori probabilities
- i.e., for every $i \in \{0, 1, \ldots, M-1\}$, we have

$$P[u_i \mid r = \rho] \quad \text{for all } \rho$$

Probability of Error

- thus, to maximize

$$P[\hat{u}(\rho) \mid r = \rho] \quad \text{for each } \rho$$

- we define $\hat{m}(\rho)$ as follows:

$$\hat{u}(\rho) = \underset{u \in \{u_0, \ldots, u_{M-1}\}}{\arg\max} P[u \mid r = \rho] \tag{7}$$

- the above rule (7) for picking $\hat{u}(\rho)$ is called the MAP rule

A Posteriori Probabilities

- a posteriori probabilities $P[u_i \mid r = \rho]$, $i = 0, \ldots, M-1$, are determined by the mixed form of Bayes rule:

$$P[u_i \mid r = \rho] = \frac{P[u_i]\ p_r(\rho \mid u_i)}{p_r(\rho)} = \frac{P[u_i]\ p_r(\rho \mid s = s_i)}{p_r(\rho)} \tag{8}$$

  where (8) follows from (2)

MAP Rule

- $p_r(\rho)$ is independent of the index i
- MAP rule (7) of the optimum receiver can be restated
- on observing $r = \rho$, set $\hat{u}(\rho) = u_k$ whenever the decision function

$$P[u_i]\, p_r(\rho \mid s = s_i);\ i = 0, 1, \ldots, M-1, \tag{9}$$

is maximum for $i = k$

ML Rule

ML Rule

- do not consider the a priori probabilities $P[u_i]$
- ML rule can be restated
- on observing $r = \rho$, set $\hat{u}(\rho) = u_k$ whenever the decision function

$$p_r(\rho \mid s = s_i);\ i = 0, 1, \ldots, M-1, \tag{10}$$

  is maximum for $i = k$
- used when a priori probabilities are not known
- ML receiver is optimal when $P[u_i] = P[u_j]$ for all $i, j \in \{0, 1, \ldots, M-1\}$

Decision Regions

Decision Regions

- recall.. the vectors are described in terms of coordinates φ_j, $j = 1, \ldots, N$
- the MAP rule [3] entails a partitioning of the entire space spanned by $\{\varphi_j\}_{j=1,\ldots,N}$ into M disjoint regions
- let $\mathcal{I}_k$ denote the kth region
- the regions are called optimum decision regions
- then MAP rule is equivalent to setting $\hat{u}(\rho) = u_k$, whenever $\rho \in \mathcal{I}_k$

[3] The discussion applies similarly to ML rule as well.