# Homework Solutions: Auction algorithm for Min-Cost Flow

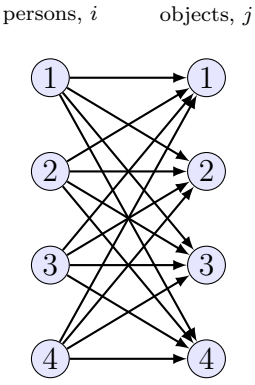
**Note:** To pass the homework, you need to reach 50% of the points.

## 1. The worst dual cost degradation of the Gauss-Seidel auction

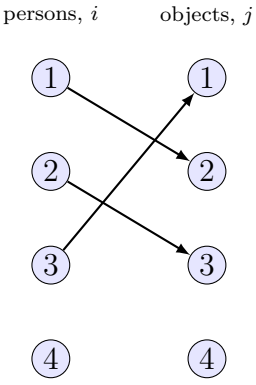
The Gauss-Seidel version of the auction algorithm resembles coordinate descent algorithms, because it involves the change of a single object price with all other prices held fixed. Such a price change may worsen strictly the value of the dual function

$$q(p) = \sum_{i=1:n} \max_{j \in A(i)} \{a_{ij} - p_j\} + \sum_{j=1:n} p_j \ .$$

This exercise guide you by an example to deduce the worst dual cost degradation. Consider the following graph representation shown above (left most figure). Suppose the cost coefficients and the current price values for objects are as given in table above. The middle figure represents the current partial assignment.




graph representation




partial assignment $\mathcal{S}$

| $a_{ij}$ | $j=1$ | $j=2$ | $j=3$ | $j=4$ |
|---|---|---|---|---|
| $i=1$ | 25 | 86 | 13 | 12 |
| $i=2$ | 43 | 30 | 74 | 25 |
| $i=3$ | 97 | 26 | 85 | 22 |
| $i=4$ | 26 | 26 | 26 | 26 |
| $p_j$ | 24 | 19 | 39 | 50 |

cost coefficients and prices

**(a)** Fix $p_1, p_3$, and $p_4$ values at their current values. Let $y_{i2}$ denote the largest value of $p_2$, below which object 2 is the best object for person $i$. Compute $y_{i2}$ for $i = 1, 2, 3, 4$. (5p)

**(b)** Arrange $y_{i2}$, $i = 1, 2, 3, 4$, in increasing order and let $z_1$ denote the smallest, $z_2$ the second smallest, $z_3$ the third smallest, and $z_4$ the largest, i.e., $z_1 \leq z_2 \leq z_3 \leq z_4$. How many persons prefer object 2 to others when 1) $p_2 \leq z_1$, 2) $p_2 \leq z_2$, 3) $p_2 \leq z_3$, 4) $p_2 \leq z_4$ and 5) $p_2 > z_4$ (5p)

**(c)** The directional derivative $d(\cdot)$ of $q(p)$ along $p_2$ at $z$ is given by

$$d(z) = 1 - (\text{number of persons prefer object 2 to others when } p_2 = z) \quad (1)$$

Use the answer of part (b) to compute the directional derivative of $q(p)$ along $p_2$. By using the directional derivative, deduce the shape of $q(24, p_2, 39, 50) = \bar{q}(p_2)$ for different ranges of $p_2$. (15p)

**(d)** Identify that $\bar{q}(p_2)$ is piece-wise linear and that the breakpoints are given by $z_i, i = 1, 2, 3, 4$. What is the range of $p_2$ that minimizes $\bar{q}(p_2)$. (5p)

**(e)** Use the sketch of $\bar{q}(p_2)$ vs $p_2$ to deduce that the worst dual cost degradation of any Gauss-Seidel auction price update is $\epsilon$. (5p)

**(f)** Verify that the partial assignment shown above (middle figure) satisfy $\epsilon$-CS for any $\epsilon > 0$. (5p)

**(g)** Carry out the steps of auction algorithm with $\epsilon = 10$ to find a feasible assignment. It is given that the optimal benefit $f^\star$ is 283. What can you say about the benefit $f^{\text{auction}}$ achieved by the auction algorithm. Verify that $f^\star - f^{\text{auction}} \leq n\epsilon$, where $n$ is the number of persons. (15p)

**Solution:**

**(a)** Consider for example the case person $i = 1$. Among the objects $j = 1, 3, 4$, person 1's best benefit is 1 achieved with $j = 1$, see the table below. If $1 \leq 86 - p_2$ (*i.e.*, $p_2 \leq 85$), then object $j = 2$ becomes preferable to person 1. Hence, $y_{12} = 85$. See the table below for other cases.

| $a_{ij} - p_j$ | $j = 1$ | $j = 2$ | $j = 3$ | $j = 4$ | |
|---|---|---|---|---|---|
| $i = 1$ | 1 | $86 - p_2$ | $-26$ | $-38$ | → best benefit = 1, $86 - p_2 = 1 \Rightarrow y_{12} = 85$ |
| $i = 2$ | 19 | $30 - p_2$ | 35 | $-25$ | → best benefit = 35, $30 - p_2 = 35 \Rightarrow y_{22} = -5$ |
| $i = 3$ | 73 | $26 - p_2$ | 46 | $-28$ | → best benefit = 73, $26 - p_2 = 73 \Rightarrow y_{32} = -47$ |
| $i = 4$ | 2 | $26 - p_2$ | -13 | $-24$ | → best benefit = 2, $26 - p_2 = 2 \Rightarrow y_{42} = 24$ |
| $p_j$ | 24 | $p_2$ | 39 | 50 | |

benefits

**(b)** We have $z_1 = -47, z_2 = -5, z_3 = 24$, and $z_4 = 85$
1) 4 2) 3 3) 2 4) 1 5) 0

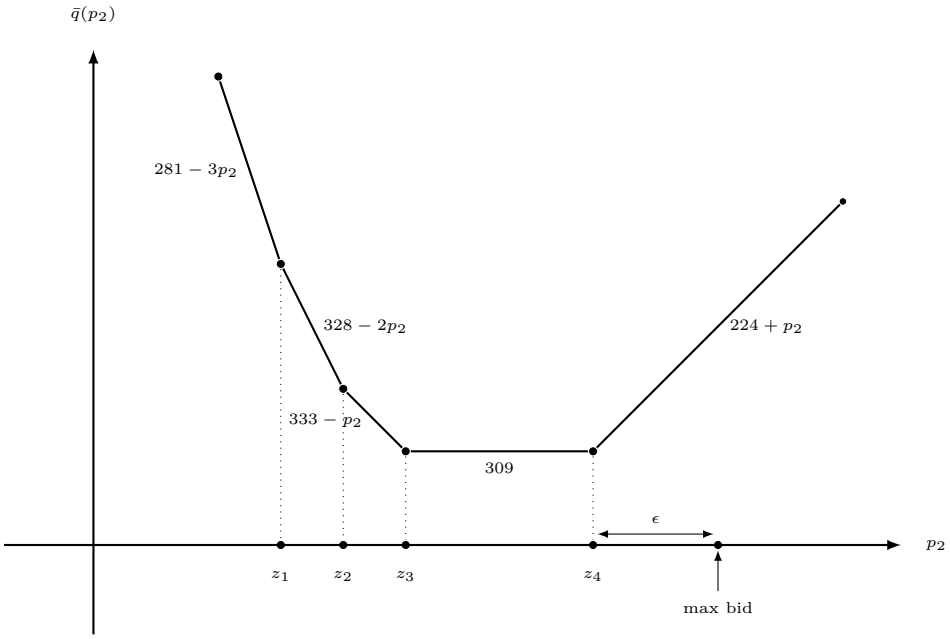
**(c)**

$$d(z) = \begin{cases} -3 & z \leq z_2 \\ -2 & z_1 < z \leq z_2 \\ -1 & z_2 < z \leq z_3 \\ 0 & z_3 < z \leq z_4 \\ 1 & z > z_4 \end{cases}$$

**(d)** By using the definition of $q(p)$ and the directional derivatives, we can compute $\bar{q}(p_2)$ as follows:

$$\bar{q}(p_2) = \begin{cases} 281 - 3p_2 & z \leq z_2 \\ 328 - 2p_2 & z_1 < z \leq z_2 \\ 333 - p_2 & z_2 < z \leq z_3 \\ 309 & z_3 < z \leq z_4 \\ 224 + p_2 & z > z_4 \ . \end{cases}$$

The range of $p_2$, $z_3 \leq p_2 \leq z_4$ minimizes $\bar{q}(p_2)$, because the directional derivative is zero in this range, see figure below.




**(e)** An update for $p_2$, (i.e., a bid for object $j = 2$) is potentially made when object 2 is preferable to at least one of the person $i = 1, 2, 3, 4$. Moreover, when $p_2 \leq y_{i2}$, person $i$'s bid is at most $y_{i2} + \epsilon$, and therefore the maximum bid is upper bounded by $\max_{i\in\{1,2,3,4\}} y_{i2} + \epsilon = z_4 + \epsilon$, see figure above. Since the slope of $\bar{q}(p_2)$ is one for $p_2 > z_4$, the worst dual cost degradation of any Gauss-Seidel auction price update is $\epsilon$

**(f)** Any person $i \in \mathcal{S}$ is assigned to the object $j_i$ for which $a_{ij} - p_j$ is maximized. Therefore, we have $a_{ij_i} - p_{j_i} = \max_{j\in\{1,2,3,4\}}\{a_{ij} - p_j\}$, which trivially implies $a_{ij_i} - p_{j_i} \geq \max_{j\in\{1,2,3,4\}}\{a_{ij} - p_j\} - \epsilon$ for any $\epsilon > 0$.

**(g)** See the table below.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $p_1$ | 24 | 24 | 24 | 49 | 61 | 61 | 61 | 61 |
| $p_2$ | 19 | 34 | 95 | 95 | 95 | 95 | 95 | 95 |
| $p_3$ | 39 | 39 | 39 | 39 | 39 | 60 | 102 | 102 |
| $p_4$ | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 71 |
| bid. | 4 | 1 | 4 | 3 | 4 | 2 | 4 | |
| obj. | 2 | 2 | 1 | 1 | 3 | 3 | 4 | |
| $\gamma_i$ | 7−2+10=15 | 52−1+10=61 | 2+(13)+10=25 | 48−46+10=12 | −13+24+10=21 | 14+18+10=42 | −24+35+10=21 | |
| ass. |  |  |  |  |  |  |  | |

We have $f^{\text{auction}} = f^\star = 283$. This clearly agrees with that $f^{\text{auction}} - f^\star = 283 = 0 < n\epsilon$.

## 2. Dealing with infeasibility

This problem considers the standard assignment problem, see p. 12 of the textbook. When implementing the auction algorithm in general, there should be a subroutine

to detect whether the problem is infeasible. One method to detect the infeasibility of the problem is to convert the problem to a feasible problem by adding a set $\bar{\mathcal{A}}$ of artificial pairs $(i,j)$ to the original set $\mathcal{A}$. Note that all artificial arc $(i,j) \in \bar{\mathcal{A}}$ should be associated with an appropriately chosen cost coefficients $a_{ij}$. Given $a_{ij} < -(2n-1)C,\ \forall (i,j) \in \bar{\mathcal{A}}$, it can be shown that

$$\begin{aligned} &\text{original problem is feasible} \\ &\quad \Rightarrow \text{no pair } (i,j) \in \bar{\mathcal{A}} \text{ participate in the optimal assignment} \\ &\qquad\qquad\qquad\qquad \text{of the artificial problem} , \end{aligned} \tag{2}$$

where $C = \max_{(i,j)\in\mathcal{A}} |a_{ij}|$.

**(a)** Discuss how to use the contrapositive of statement (2) to detect infeasibility of the original problem. (10p)

**(b)** To prove this claim by contradiction, suppose the optimal assignment $\mathcal{S}^\star$ to the artificial problem contains at least one artificial pair $(i,j) \in \bar{\mathcal{A}}$. We denote by $\bar{\mathcal{S}}$ the *nonempty* subset of artificial pairs in $\mathcal{S}^\star$. Then we have

$$\sum_{(i,j)\in\bar{\mathcal{S}}} a_{ij} + \sum_{(i,j)\in\mathcal{S}^\star\setminus\bar{\mathcal{S}}} a_{ij} \geq \sum_{(i,j)\in\mathcal{S}} a_{ij} \ ,\ \forall \text{ feasible assignment } \mathcal{S} \ . \tag{3}$$

Argue why the above statement (3) is true. (10p)

**(c)** From statement (3) above, deduce that

$$\begin{aligned} \sum_{(i,j)\in\bar{\mathcal{S}}} a_{ij} \ &\geq \sum_{(i,j)\in\mathcal{S}} a_{ij} - \sum_{(i,j)\in\mathcal{S}^\star\setminus\bar{\mathcal{S}}} a_{ij} \ ,\ \forall \text{ feasible assignment } \mathcal{S} \\ &\geq -(2n-1)C \ . \end{aligned} \tag{4}$$

Argue why the second inequality above holds. (15p)

**(d)** By using the claim of part (c) above, identify a contradiction to conclude the proof of (2). (10p)

**Solution:**

**(a)** contrapositive: Given $a_{ij} < -(2n-1)C,\ \forall (i,j) \in \bar{\mathcal{A}}$,

$$\begin{aligned} &\text{at least one pair } (i,j) \in \bar{\mathcal{A}} \text{ participate in the optimal assignment} \\ &\qquad\qquad \text{of the artificial problem} \Rightarrow \text{original problem is infeasible} . \end{aligned}$$

**(b)** Suppose (3) is *wrong*. This implies that $\exists$ a feasible $\mathcal{S}$ such that $\sum_{(i,j)\in\bar{\mathcal{S}}} a_{ij} + \sum_{(i,j)\in\mathcal{S}^\star\setminus\bar{\mathcal{S}}} a_{ij} < \sum_{(i,j)\in\mathcal{S}} a_{ij}$, which is a *contradiction*.

**(c)** We have $\sum_{(i,j)\in\mathcal{S}} a_{ij} \geq -nC$, because $\mathcal{S}$ has $n$ elements and $a_{ij} \geq -C$. Note that $\bar{\mathcal{S}}$ is *nonempty*. Therefore, we have $-\sum_{(i,j)\in\mathcal{S}^\star\setminus\bar{\mathcal{S}}} a_{ij} \geq -(n-1)C$, because $\mathcal{S}$ has at most $(n-1)$ elements and $a_{ij} \geq -C$. Combining these concludes the claim.

**(d)** statement (4) contradicts $a_{ij} < -(2n-1)$, which was set from the beginning.