Homework: Auction algorithm for Min-Cost Flow

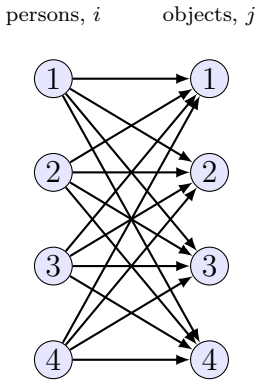
Note: To pass the homework, you need to reach 50% of the points.

1. The worst dual cost degradation of the Gauss-Seidel auction

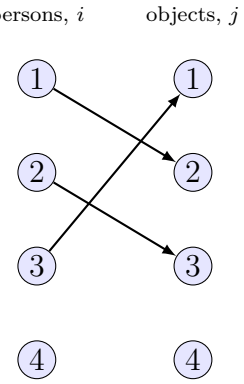
The Gauss-Seidel version of the auction algorithm resembles coordinate descent algorithms, because it involves the change of a single object price with all other prices held fixed. Such a price change may worsen strictly the value of the dual function

$$q(p) = \sum_{i=1:n} \max_{j \in A(i)} \{a_{ij} - p_j\} + \sum_{j=1:n} p_j \ .$$

This exercise guide you by an example to deduce the worst dual cost degradation. Consider the following graph representation shown above (left most figure). Suppose the cost coefficients and the current price values for objects are as given in table above. The middle figure represents the current partial assignment.



graph representation



partial assignment $\mathcal{S}$

a_{ij}	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	25	86	13	12
$i=2$	43	30	74	25
$i=3$	97	26	85	22
$i=4$	26	26	26	26
p_j	24	19	39	50

cost coefficients and prices

(a) Fix p_1, p_3, and p_4 values at their current values. Let y_{i2} denote the largest value of p_2, below which object 2 is the best object for person i. Compute y_{i2} for $i = 1, 2, 3, 4$. (5p)

(b) Arrange y_{i2}, $i = 1, 2, 3, 4$, in increasing order and let z_1 denote the smallest, z_2 the second smallest, z_3 the third smallest, and z_4 the largest, i.e., $z_1 \leq z_2 \leq z_3 \leq z_4$. How many persons prefer object 2 to others when 1) $p_2 < z_1$, 2) $p_2 < z_2$, 3) $p_2 < z_3$, and 4) $p_2 > z_2$ (5p)

(c) The directional derivative $d(\cdot)$ of $q(p)$ along p_2 at z is given by

$$d(z) = 1 - (\text{number of persons prefer object 2 to others when } p_2 = z) \quad (1)$$

Use the answer of part (b) to compute the directional derivative of $q(p)$ along p_2. By using the directional derivative, deduce the shape of $q(24, p_2, 39, 50) = \bar{q}(p_2)$ for different ranges of p_2. (15p)

(d) Identify that $\bar{q}(p_2)$ is piece-wise linear and that the breakpoints are given by $z_i, i = 1, 2, 3, 4$. What is the range of p_2 that minimizes $\bar{q}(p_2)$. (5p)

(e) Use the sketch of $\bar{q}(p_2)$ vs p_2 to deduce that the worst dual cost degradation of any Gauss-Seidel auction price update is ϵ. (5p)

(f) Verify that the partial assignment shown above (middle figure) satisfy ϵ-CS for any $\epsilon > 0$. (5p)

(g) Carry out the steps of auction algorithm with $\epsilon = 10$ to find a feasible assignment. It is given that the optimal benefit $f^\star$ is 283. What can you say about the benefit f^{auction} achieved by the auction algorithm. Verify that $f^\star - f^{\text{auction}} \leq n\epsilon$, where n is the number of persons. (15p)

2. Dealing with infeasibility

This problem considers the standard assignment problem, see p. 12 of the textbook. When implementing the auction algorithm in general, there should be a subroutine to detect whether the problem is infeasible. One method to detect the infeasibility of the problem is to convert the problem to a feasible problem by adding a set $\bar{\mathcal{A}}$ of artificial pairs (i, j) to the original set $\mathcal{A}$. Note that all artificial arc $(i, j) \in \bar{\mathcal{A}}$ should be associated with an appropriately chosen cost coefficients a_{ij}. Given $a_{ij} < -(2n-1)C, \ \forall (i,j) \in \bar{\mathcal{A}}$, it can be shown that

$$\begin{aligned} &\text{original problem is feasible} \\ &\quad \Rightarrow \text{no pair } (i,j) \in \bar{\mathcal{A}} \text{ participate in the optimal assignment} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{of the artificial problem} \ , \end{aligned} \tag{2}$$

where $C = \max_{(i,j)\in\mathcal{A}} |a_{ij}|$.

(a) Discuss how to use the contrapositive of statement (2) to detect infeasibility of the original problem. (10p)

(b) To prove this claim by contradiction, suppose the optimal assignment $\mathcal{S}^\star$ to the artificial problem contains at least one artificial pair $(i, j) \in \bar{\mathcal{A}}$. We denote by $\bar{\mathcal{S}}$ the *nonempty* subset of artificial pairs in $\mathcal{S}^\star$. Then we have

$$\sum_{(i,j)\in\bar{\mathcal{S}}} a_{ij} + \sum_{(i,j)\in\mathcal{S}^\star\setminus\bar{\mathcal{S}}} a_{ij} \geq \sum_{(i,j)\in\mathcal{S}} a_{ij} \ , \ \forall \text{ feasible assignment } \mathcal{S} \ . \tag{3}$$

Argue why the above statement (3) is true. (10p)

(c) From statement (3) above, deduce that

$$\begin{aligned} \sum_{(i,j)\in\bar{\mathcal{S}}} a_{ij} &\geq \sum_{(i,j)\in\mathcal{S}} a_{ij} - \sum_{(i,j)\in\mathcal{S}^\star\setminus\bar{\mathcal{S}}} a_{ij} \ , \ \forall \text{ feasible assignment } \mathcal{S} \\ &\geq -(2n-1)C \ . \end{aligned} \tag{4}$$

Argue why the second inequality above holds. (15p)

(d) By using the claim of part (b) above, identify a contradiction to conclude the proof of (2). (10p)