L4: The Min-Cost Flow Problem

FEL3250: Network Optimization

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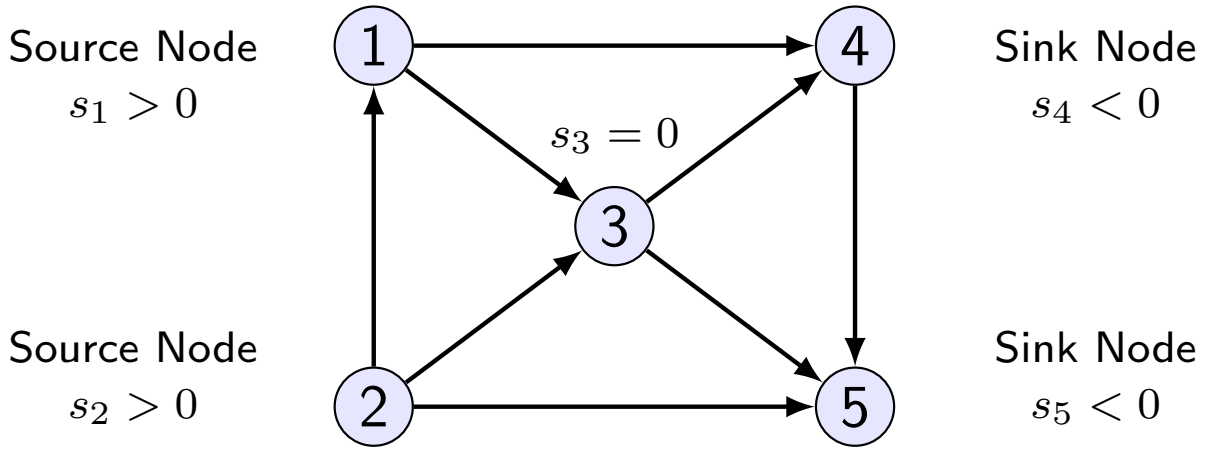
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Outline

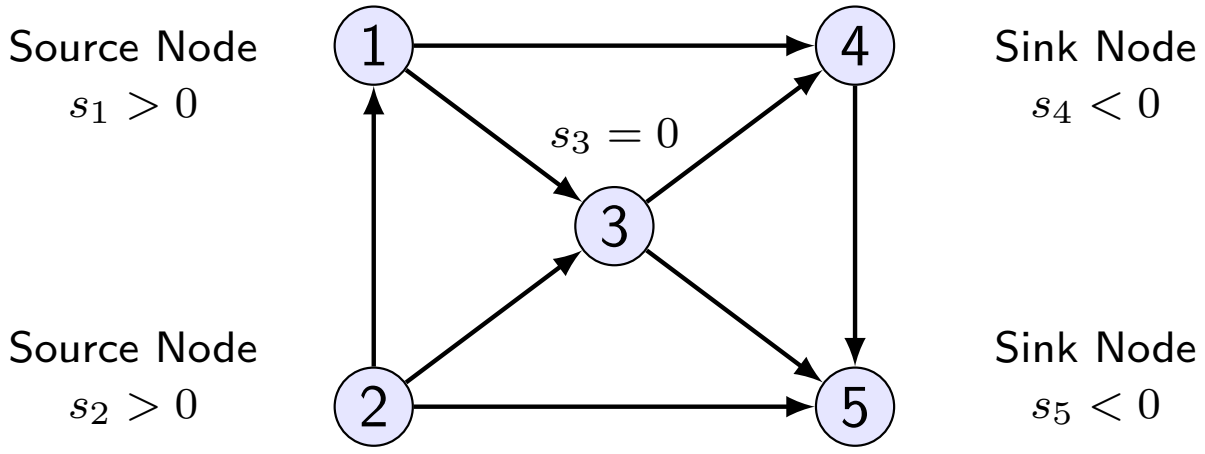
- Introduction (brief)
- Equivalent formulations of `Min-Cost Flow` problem
- Duality

Introduction
(Brief)

The Min-Cost Flow Problem

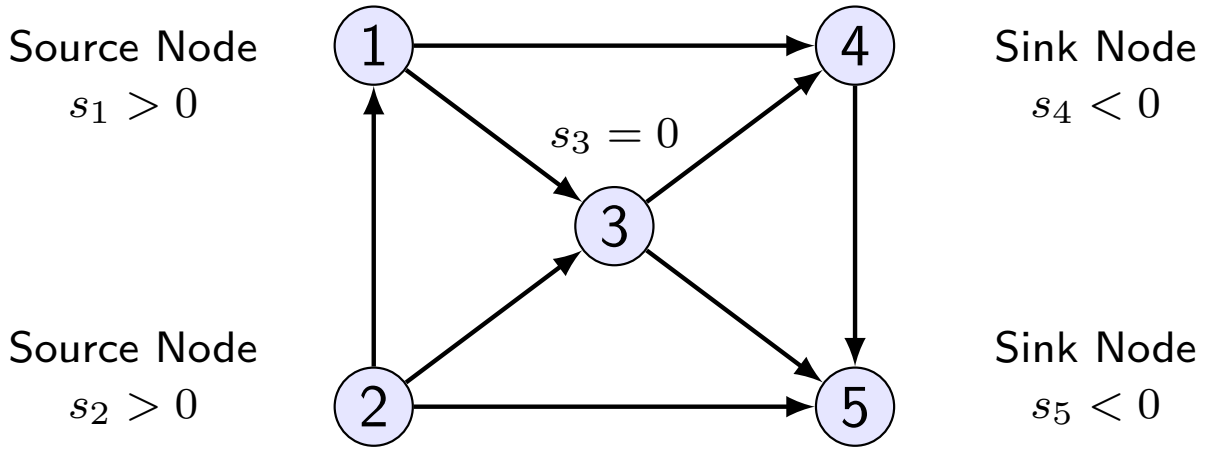


The Min-Cost Flow Problem (example)



- node: plants or warehouses, indexed by i
- s_i: # of cars produced ($s_i > 0$) or stored ($s_i < 0$) in node i
- arcs: routes
- a_{ij}: transportation cost per car
- `Min-Cost Flow` problem finds the # of cars to be scheduled in every shipping route to minimize the total cost

The Min-Cost Flow Problem (in general)



- applications are well BEYOND its simple use
- `Shortest Path` problem
- `Assignment` problem
- `Max Flow` problem
- `Transportation` problem

- different formulations $\Rightarrow$ different benefits

Equivalent formulations

lend themselves better or worse for various analytical and computational purposes

Equivalent Formulations

Setting the Lower Flow Bounds to Zero

- $$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i, \ \forall\, i \in \mathcal{N} \\ & b_{ij} \le x_{ij} \le c_{ij}, \ \forall\, (i,j) \in \mathcal{A}\ , \end{aligned}$$

- use variable transformation: $y_{ij} = x_{ij} - b_{ij}$
- discuss ...

Equivalent Formulations

Setting the Lower Flow Bounds to Zero

Equivalent Formulations

Eliminating the Upper Flow Bounds

- introduce new variables: $z_{ij} \geq 0$
- $x_{ij} + z_{ij} = c_{ij}$
- discuss ...

Equivalent Formulations

Eliminating the Upper Flow Bounds

Equivalent Formulations

Reduction to a Circulation Format

- introduce an accumulation node t with *tight* capacity constraints OR
- introduce an accumulation node t with usual capacity constraints and very small arc costs
- discuss ...

Equivalent Formulations

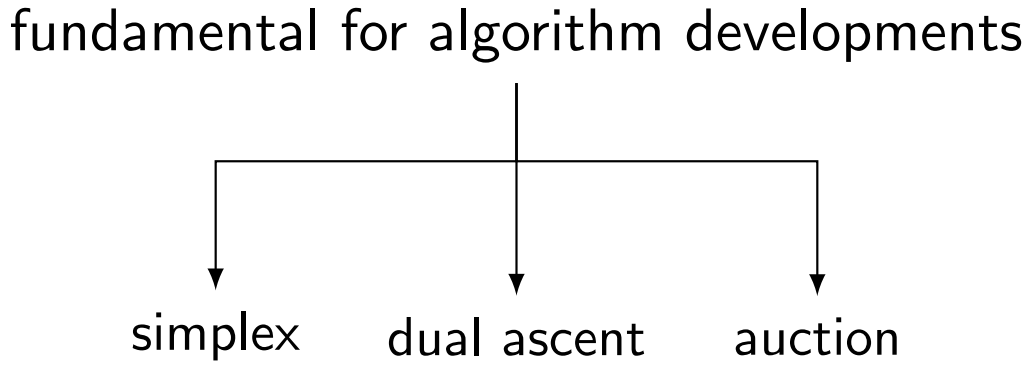
Reduction to a Circulation Format

Equivalent Formulations

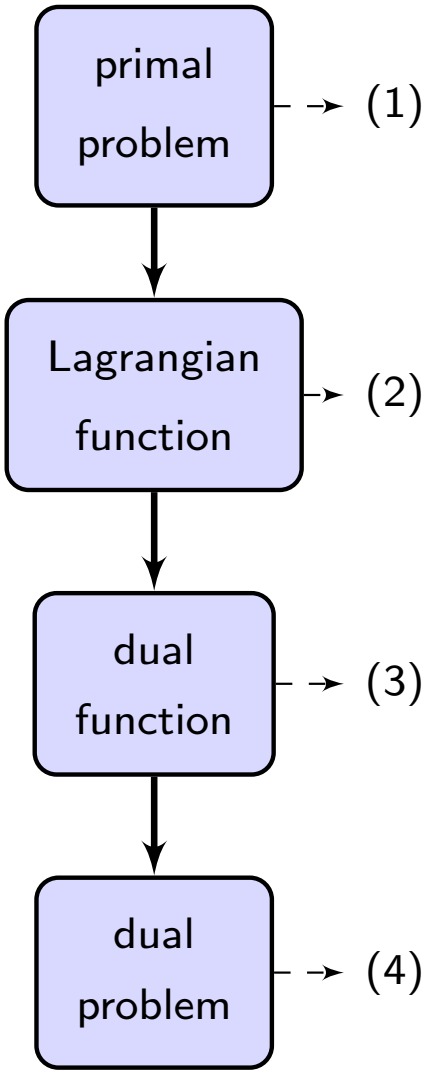
Reduction to an Assignment Problem

- first transform to a transportation problem
- then transform to an assignment problem
- try yourself.

Duality


fundamental for algorithm developments
simplex
dual ascent
auction

Dual problem



minimize $f(x)$
subject to $h_i(x) = 0,\ i = 1, 2$
$x \in \mathcal{X}$

Dual problem

(1)– `Min-Cost Flow` problem

$$\begin{aligned}
\text{minimize} \quad & f(x) = \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i, \ \forall\ i \in \mathcal{N} \\
& b_{ij} \leq x_{ij} \leq c_{ij}, \ \forall\ (i,j) \in \mathcal{A}\ ,
\end{aligned}$$

(2)– partial Lagrangian function

$$L(x,p) =$$

$$=$$

Dual problem

(3)– dual function

$$q(p) = \min_{b_{ij} \le x_{ij} \le c_{ij}, (i,j) \in \mathcal{A}} L(x,p)$$

$$=$$

$$=$$

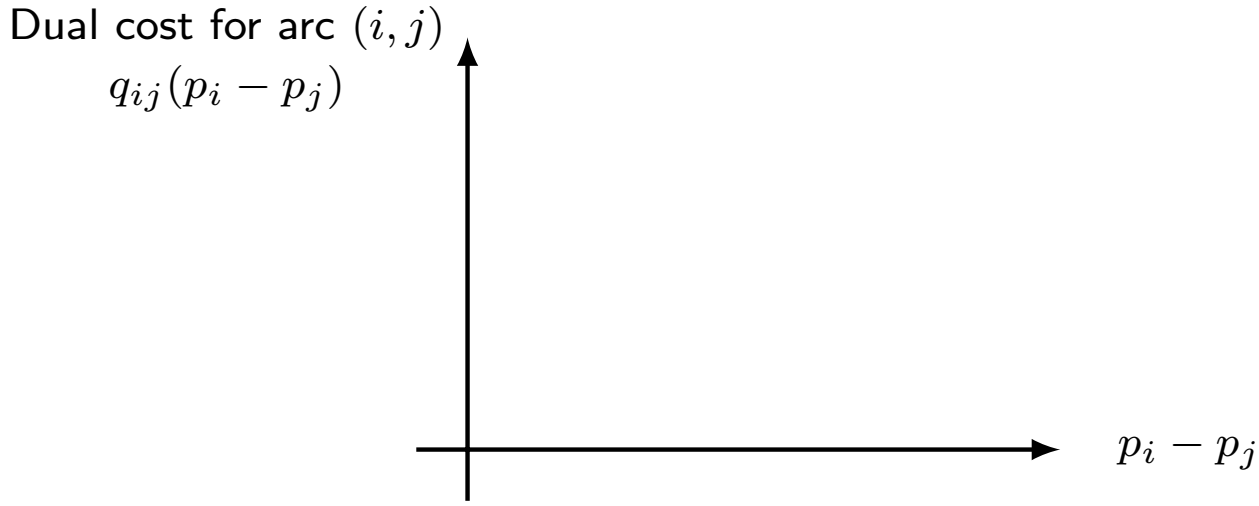
$$=$$

$$q_{ij}(p_i - p_j) = \begin{cases} (a_{ij} + p_j - p_i) b_{ij} & p_i - p_j \le a_{ij} \\ (a_{ij} + p_j - p_i) c_{ij} & p_i - p_j > a_{ij} \end{cases}$$

Dual problem

(3)– dual function (cont.)

$$q_{ij}(p_i - p_j) = \begin{cases} a_{ij}b_{ij} - b_{ij}(p_i - p_j) & p_i - p_j \leq a_{ij} \\ a_{ij}c_{ij} - c_{ij}(p_i - p_j) & p_i - p_j > a_{ij} \end{cases}$$



(4)– dual problem

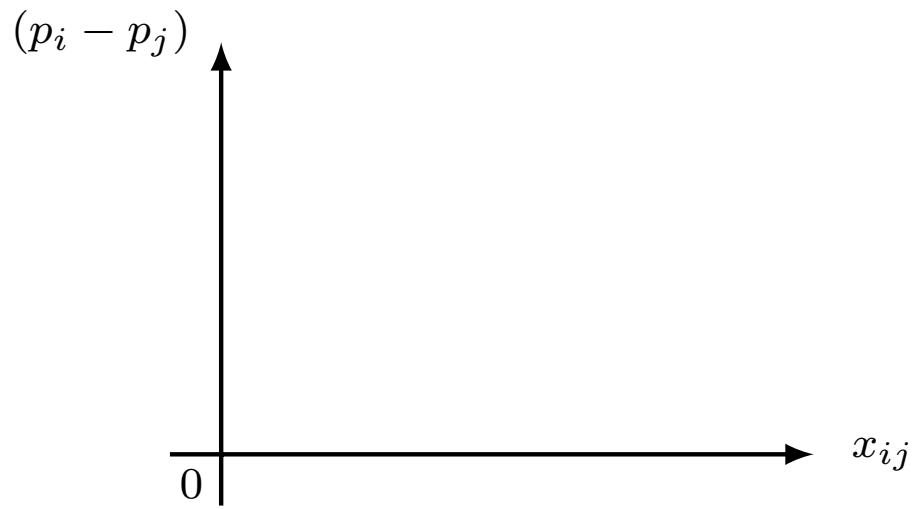
$$\text{maximize} \quad q(p) = \sum_{(i,j)\in\mathcal{A}} q_{ij}(p_i - p_j) + \sum_{i\in\mathcal{N}} s_i p_i$$

Complementary slackness (CS)

Definition

Flow-price vector (x, p) satisfies CS condition if and only if

$$x_{ij} = \underset{b_{ij} \le z_{ij} \le c_{ij}}{\text{argmin}} \quad (a_{ij} + p_j - p_i) z_{ij}$$



Optimality

Proposition

A feasible flow vector $x^\star$ and a price vector $p^\star$ satisfy CS *iff* $x^\star, p^\star$ are primal, dual optimal AND $f(x^\star) = q(p^\star)$.

Optimality

proof ($\Longrightarrow$)

$$
\begin{aligned}
q(p) &= \min_{b_{ij} \le x_{ij} \le c_{ij},\ (i,j)\in\mathcal{A}} L(x,p) \\
&\le \\
&\le \\
&= \\
&= \sum_{(i,j)\in\mathcal{A}} a_{ij} x_{ij}, \quad \forall \text{ feasible } x
\end{aligned}
$$

We also have

$$
\begin{aligned}
q(p^\star) &= \\
&= \\
&= \sum_{(i,j)\in\mathcal{A}} a_{ij} x^\star_{ij}
\end{aligned}
$$

Optimality

proof ($\Longleftarrow$)

$$f(x^\star) = q(p^\star)$$

$$\Longrightarrow L(x^\star, p^\star) = \min_{b_{ij} \le z_{ij} \le c_{ij},\ (i,j)\in\mathcal{A}} L(z, p^\star)$$

$$\Longrightarrow$$

$$\Longrightarrow x^\star_{ij} = \operatorname*{argmin}_{b_{ij} \le z_{ij} \le c_{ij}} (a_{ij} + p^\star_j - p^\star_i) z_{ij}$$

Summary

- **many application domains:** communication, transportation, manufacturing, modeling important class of combinatorial problems, including assignment, shortest path, among others
- **equivalent formulations:** lend themselves better or worse for various analytical and computational purposes
- **duality theory:** fundamental for algorithm developments, e.g.,
  - simplex methods
  - dual ascent methods
  - auction algorithms (next lecture)

Thank You!

Questions?

For more questions:

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