Homework Solutions: The Min-Cost Flow Problem

1. Reduction to One Source/One Sink Format

Show how the minimum cost flow problem can be transformed to an equivalent problem where all node supplies are zero except for one node that has positive supply and one node that has negative supply. (10p)

Solution:

The original problem is [compare with Figure 1(a)].

$$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i, \ \forall\ i \in \mathcal{N} \\ & 0 \le x_{ij} \le c_{ij}, \ \forall\ (i,j) \in \mathcal{A}\ . \end{aligned}$$

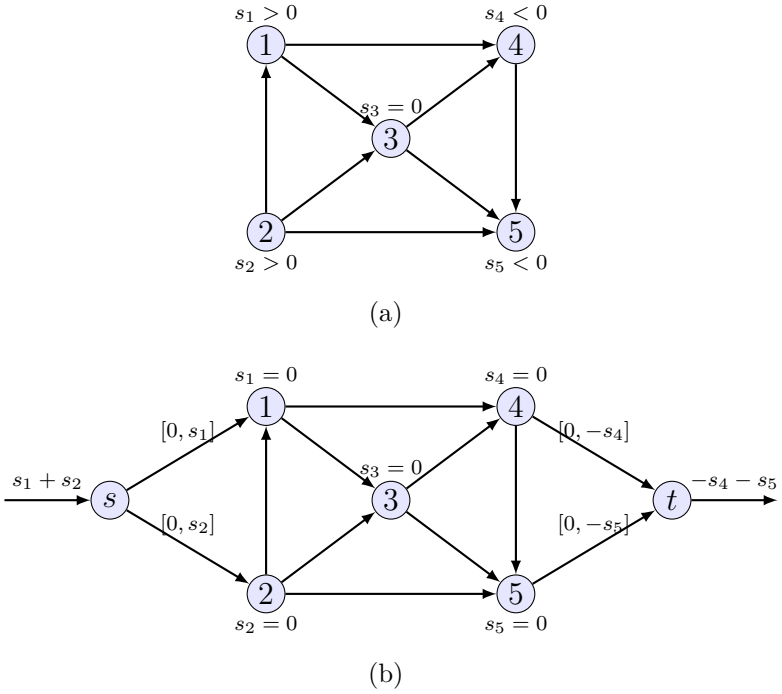


Figure 1: (a) Original network; (b) Modified equivalent network.

Let s denote the node with positive supply and t denote the node with negative supply. Next introduce an arc (s, i) and a constraint $0 \le x_{si} \le s_i$ for all i with $s_i > 0$ and an arc (i, t) and a constraint $0 \le x_{it} \le -s_i$ for all i with $s_i < 0$. Then

the modified equivalent problem is [compare with Figure 1(b)].

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = \begin{cases} x_{si} & \forall\ i\in\mathcal{N} \text{ with } s_i>0 \\ -x_{it} & \forall\ i\in\mathcal{N} \text{ with } s_i<0 \\ 0 & \text{otherwise .} \end{cases} \\
& \sum_{\{i|i\in\mathcal{N},s_i>0\}} x_{si} = \sum_{\{i|i\in\mathcal{N},s_i>0\}} s_i \\
& \sum_{\{i|i\in\mathcal{N},s_i<0\}} x_{it} = \sum_{\{i|i\in\mathcal{N},s_i<0\}} s_i \\
& 0\le x_{ij}\le c_{ij},\ \forall\ (i,j)\in\mathcal{A} \\
& 0\le x_{si}\le s_i,\ \forall\ i\in\mathcal{N} \text{ with } s_i>0 \\
& 0\le x_{it}\le -s_i,\ \forall\ i\in\mathcal{N} \text{ with } s_i<0\ .
\end{aligned}
$$

2. Duality for Nonnegativity Constraints

Consider the following minimum cost flow problem where there are nonnegativity constraints

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i,\ \forall\ i\in\mathcal{N} \\
& 0\le x_{ij},\ \forall\ (i,j)\in\mathcal{A}\ .
\end{aligned}
$$

(a) Show that a dual function of the Min-Cost Flow problem above is given by

$$
q(p) = \begin{cases} \sum_{i\in\mathcal{N}} s_i p_i & p_i - p_j \le a_{ij},\ \forall\ (i,j)\in\mathcal{A} \\ -\infty & \text{otherwise,} \end{cases}
$$

and the corresponding dual problem is given by

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i\in\mathcal{N}} s_i p_i \\
\text{subject to} \quad & p_i - p_j \le a_{ij},\ \forall\ (i,j)\in\mathcal{A},
\end{aligned}
$$

with variable p. (10p)

(b) Show that a feasible flow vector $x^\star$ and a price vector $p^\star$ satisfy the following `CS` condition

$$
\begin{aligned}
& p_i^\star - p_j^\star \le a_{ij}, \quad \forall\ (i,j)\in\mathcal{A}\ , \\
& p_i^\star - p_j^\star = a_{ij}, \quad \forall\ (i,j)\in\mathcal{A} \text{ with } 0<x_{ij}^\star.
\end{aligned}
$$

if and only if $x^\star$ is primal optimal, $p^\star$ is an optimal solution of the dual problem in part (a) above, and the optimal primal and dual costs are equal. (10p)

Solution: The solution to this problem is very similar to the one we discussed at the lecture. See page 167 of the textbook.

3. Linear Programming with Consecutive Ones in Columns

Consider the following linear program (LP)

$$\begin{aligned} &\text{minimize} && c^{\mathrm{T}}x \\ &\text{subject to} && \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}}_{B} x \geq \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b} \\ &&& x \geq 0 \ . \end{aligned} \tag{1}$$

(a) Introduce the surplus variable $y_i \geq 0$ for ith row ($i = 1, 2, 3$) of the constraint $Bx \geq b$ above. Let $y = [y_1, y_2, y_3]^{\mathrm{T}}$. Now the inequality constraint $Bx \geq b$ can be converted to a form

$$\underbrace{\left[\begin{array}{c|c} B & \bar{B} \end{array}\right]}_{C} \begin{bmatrix} x \\ y \end{bmatrix} = b \ . \tag{2}$$

Compute $\bar{B}$. (5p)

(b) Consider the equivalent linear system

$$\begin{bmatrix} C \\ 0^{\mathrm{T}} \end{bmatrix}_{4\times 9} \begin{bmatrix} x \\ y \end{bmatrix}_{9\times 1} = \begin{bmatrix} b \\ 0 \end{bmatrix}_{4\times 1} \ . \tag{3}$$

Perform the following elementary row operation for $i = 3, 2, 1$, in the stated order: subtract the ith constraint of the linear system above from its $(i+1)$th constraint and use the result to replace $(i+1)$th constraint. Let the resultant linear system be

$$\bar{A} \begin{bmatrix} x \\ y \end{bmatrix} = s \ . \tag{4}$$

Compute $\bar{A}$ and s. List properties of $\bar{A}$ and s, one for each. (10p)

(c) Argue that the following LP

$$\begin{aligned} &\text{minimize} && c^{\mathrm{T}}x \\ &\text{subject to} && \bar{A} \begin{bmatrix} x \\ y \end{bmatrix} = s \\ &&& x \geq 0, \ y \geq 0 \ , \end{aligned} \tag{5}$$

is equivalent to the original LP (1). (5p)

(d) By using a graph representation, show that the LP (5) can be posed as a Min-Cost Flow problem. (*Hint:* Every row of $\bar{A}$ can be associated with a node in the graph. Positive (or negative) entries of row i can be associated with outgoing (respectively, incoming) arcs of node i.) (15p)

Solution:

(a)

$$\bar{B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) The equivalent linear system is

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ 0 \end{array}\right] .$$

We have to perform 3 elementary row operations. After the first one, we have

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ -b_3 \end{array}\right] .$$

After the second elementary operation, we have

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 - b_2 \\ -b_3 \end{array}\right] .$$

After the third elementary operation, we have

$$\underbrace{\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{\bar{A}} \left[\begin{array}{c} x \\ y \end{array}\right] = \underbrace{\left[\begin{array}{c} b_1 \\ b_2 - b_1 \\ b_3 - b_2 \\ -b_3 \end{array}\right]}_{s} .$$

The following properties of $\bar{A}$ and s are noted: every column of $\bar{A}$ has two non-zero elements, (in particular 1 and -1) and the sum of the elements of s is zero.

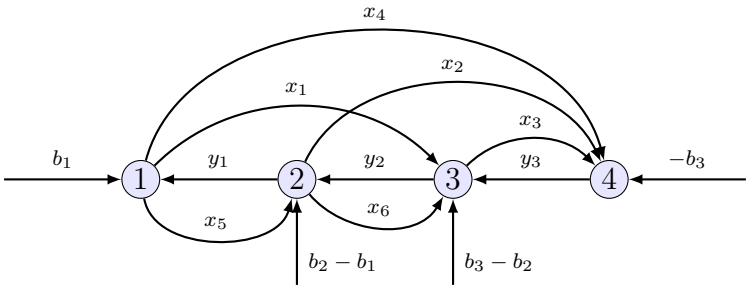
(c) It is not difficult to show that the introduction of surplus variables results in an equivalent formulation for the original problem (1). That is the original problem (1) is equivalent to

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}} x \\ \text{subject to} & C \left[\begin{array}{c} x \\ y \end{array}\right] = b \\ & x \geq 0,\ y \geq 0 \end{array} \tag{6}$$

with variables x and y. In addition, we have

$$C \left[\begin{array}{c} x \\ y \end{array}\right] = b \iff \left[\begin{array}{c} C \\ 0^{\mathrm{T}} \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} b \\ 0 \end{array}\right] \iff \bar{A} \left[\begin{array}{c} x \\ y \end{array}\right] = s \ . \tag{7}$$

The first equivalence above is trivial. The second equivalence follows from that $\bar{A} = E[C^{\mathrm{T}}\ 0]^{\mathrm{T}}$ and $s = E[b^{\mathrm{T}}\ 0]^{\mathrm{T}}$ for some *invertible* matrix E. Thus, from (6) and (7) it follows that the original problem (1) is equivalent to LP (5).



(d) The graph representation is shown above: By using the graph representation, the corresponding Min-Cost-Flow problem can be written in a straightforward manner.

4. Vehicle Fleet Planning

A Company uses a large fleet of vehicles which it leases from manufacturers. The company has forecast the following pattern of vehicle requirements for the next 3 months:

Month	Jan.	Feb.	Mar.
Vehicle requirement	430	410	440

The company can lease vehicles from several manufacturers at various costs and for various lengths of time. Two of the plans appear to be the best available:

(A) 2-month lease for \$1700 per vehicle. The company can undertake such a lease at the beginning of any month.
(B) 3-month lease for \$2200 per vehicle. The company can undertake the lease at the beginning of January only.

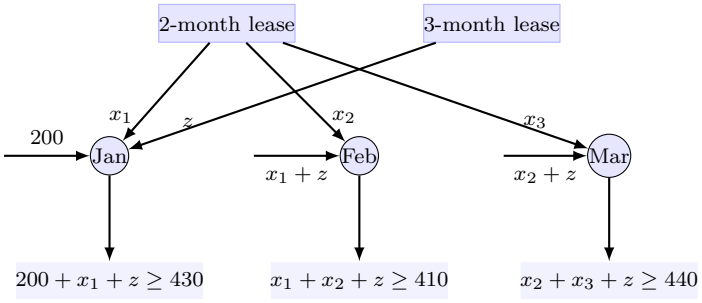
On January 1 the company has 200 cars already on lease, all of which go off lease at the end of January itself.

Formulate the problem of determining the most economical leasing policy as a minimum cost flow problem. (*Hint*: Observe that the linear (integer) programming formulation of this problem has consecutive l's in each column. Then use the result in Exercise 3.) (35p)

Solution:

Let x_i denote the number of cars under a 2-month lease at the beginning of month i ($i = 1, 2, 3$), y denote the number of cars under the 3-month lease at the beginning of month 1, $x = (x_1, x_2, x_3, z)$, and $c = (1700, 1700, 1700, 2200)$.

The figure below, schematically illustrate the problem, which is given by

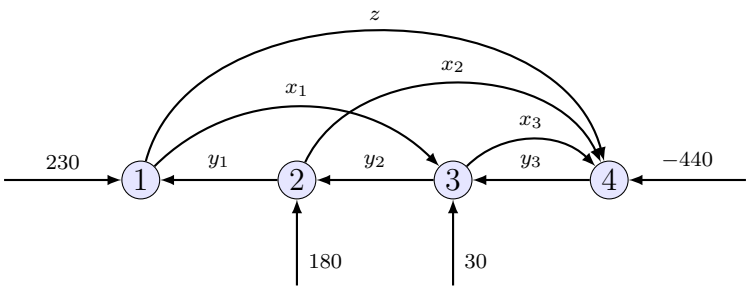


$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}} x \\ \text{subject to} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} x \geq \begin{bmatrix} 230 \\ 410 \\ 440 \end{bmatrix} \\ & x \geq 0 \ . \end{array} \tag{8}$$

By following the method of Exercise 3, we get the following equivalent formulation:

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}} x \\ \text{subject to} & \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 230 \\ 180 \\ 30 \\ -440 \end{bmatrix} \\ & x \geq 0 \ , y \geq 0 \ . \end{array} \tag{9}$$

The problem above is an equivalent Min-Cost-Flow problem for (8). The graph representation is shown below. The associated optimal cost is 863000\$, which can



be achieved by many solutions, e.g., 1) $x = (x_1, x_2, x_3, z) = (0, 191, 19, 230)$, $y = (0, 11, 0)$; 2) $x = (0, 210, 0, 230)$, $y = (0, 30, 0)$.