Homework: The Min-Cost Flow Problem

Note: To pass the homework, you need to reach 50% of the points.

1. Reduction to One Source/One Sink Format

Show how the minimum cost flow problem can be transformed to an equivalent problem where all node supplies are zero except for one node that has positive supply and one node that has negative supply. (10p)

2. Duality for Nonnegativity Constraints

Consider the following minimum cost flow problem where there are nonnegativity constraints

$$\begin{array}{ll} \text{minimize} & \sum\limits_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\ \text{subject to} & \sum\limits_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum\limits_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i,\ \forall\ i \in \mathcal{N} \\ & 0 \le x_{ij},\ \forall\ (i,j) \in \mathcal{A}\ . \end{array}$$

(a) Show that a dual function of the Min-Cost Flow problem above is given by

$$q(p) = \begin{cases} \sum\limits_{i\in\mathcal{N}} s_i p_i & p_i - p_j \le a_{ij},\ \forall\ (i,j) \in \mathcal{A} \\ -\infty & \text{otherwise,} \end{cases}$$

and the corresponding dual problem is given by

$$\begin{array}{ll} \text{maximize} & \sum\limits_{i\in\mathcal{N}} s_i p_i \\ \text{subject to} & p_i - p_j \le a_{ij},\ \forall\ (i,j) \in \mathcal{A}, \end{array}$$

with variable p. (10p)

(b) Show that a feasible flow vector $x^\star$ and a price vector $p^\star$ satisfy the following `CS` condition

$$\begin{aligned} &p_i^\star - p_j^\star \le a_{ij}, \quad \forall\ (i,j) \in \mathcal{A}\ , \\ &p_i^\star - p_j^\star = a_{ij}, \quad \forall\ (i,j) \in \mathcal{A} \text{ with } 0 < x_{ij}^\star. \end{aligned}$$

if and only if $x^\star$ is primal optimal, $p^\star$ is an optimal solution of the dual problem in part (a) above, and the optimal primal and dual costs are equal. (10p)

3. Linear Programming with Consecutive Ones in Columns

Consider the following linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}}_{B} x \geq \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b} \\ & x \geq 0 \ . \end{array} \tag{1}$$

(a) Introduce the surplus variable $y_i \geq 0$ for ith row ($i = 1, 2, 3$) of the constraint $Bx \geq b$ above. Let $y = [y_1, y_2, y_3]^{\mathrm{T}}$. Now the inequality constraint $Bx \geq b$ can be converted to a form

$$\underbrace{\left[B \;\middle|\; \bar{B}\right]}_{C} \begin{bmatrix} x \\ y \end{bmatrix} = b \ . \tag{2}$$

Compute $\bar{B}$. (5p)

(b) Consider the equivalent linear system

$$\begin{bmatrix} C \\ 0^{\mathrm{T}} \end{bmatrix}_{4\times 9} \begin{bmatrix} x \\ y \end{bmatrix}_{9\times 1} = \begin{bmatrix} b \\ 0 \end{bmatrix}_{4\times 1} \ . \tag{3}$$

Perform the following elementary row operation for $i = 3, 2, 1$, in the stated order: subtract the ith constraint of the linear system above from its $(i+1)$th constraint and use the result to replace $(i+1)$th constraint. Let the resultant linear system be

$$\bar{A} \begin{bmatrix} x \\ y \end{bmatrix} = s \ . \tag{4}$$

Compute $\bar{A}$ and s. List properties of $\bar{A}$ and s, one for each. (10p)

(c) Argue that the following LP

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & \bar{A} \begin{bmatrix} x \\ y \end{bmatrix} = s \\ & x \geq 0, \ y \geq 0 \ , \end{array} \tag{5}$$

is equivalent to the original LP (1). (5p)

(d) By using a graph representation, show that the LP (5) can be posed as a Min-Cost Flow problem. (*Hint:* Every row of $\bar{A}$ can be associated with a node in the graph. Positive (or negative) entries of row i can be associated with outgoing (respectively, incoming) arcs of node i.) (15p)

4. Vehicle Fleet Planning

A Company uses a large fleet of vehicles which it leases from manufacturers. The company has forecast the following pattern of vehicle requirements for the next 3 months:

Month	Jan.	Feb.	Mar.
Vehicle requirement	430	410	440

The company can lease vehicles from several manufacturers at various costs and for various lengths of time. Two of the plans appear to be the best available:

(A) 2-month lease for \$1700 per vehicle. The company can undertake such a lease at the beginning of any month.
(B) 3-month lease for \$2200 per vehicle. The company can undertake the lease at the beginning of January only.

On January 1 the company has 200 cars already on lease, all of which go off lease at the end of January itself.

Formulate the problem of determining the most economical leasing policy as a minimum cost flow problem. (*Hint*: Observe that the linear (integer) programming formulation of this problem has consecutive l's in each column. Then use the result in Exercise 3.) (35p)