Network Optimization: Continuous and Discrete Models

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Introduction

Network optimization problems (NOP) arise naturally in the analysis and design of large systems

How to select models that minimize the cost of transfer from supply to demand

There are four main classes of NOP:

(1) Transhipment or minimum cost ow problem Single commodity and linear cost function Shortest path, Max flow, assignment,transportation problems
(2) Single commodity network ow problem with convex cost
(3) Multicommodity network ow problem with linear or convex cost.
(4) Discrete network optimization, quantities transferred along the routes of the network are restricted to take one of a finite number of values.

The focus of the course is on modelling+analysis+solution methods+(graphics)

Graph and Flow

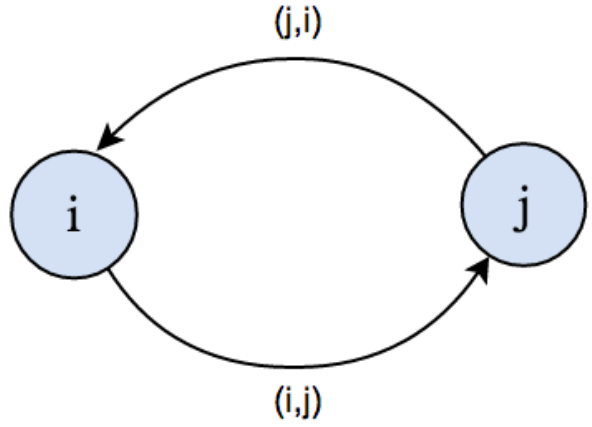
$$\mathcal{G} = (\mathcal{N}, \mathcal{A}) \quad \textit{directed graph}$$

where $\mathcal{N}$ is set of nodes, $\mathcal{A}$ is set of pairs of distinct nodes of $\mathcal{N}$ called arcs

- $|\mathcal{N}| = N$, Number of nodes
- $|\mathcal{A}| = A$, Number of arcs

(i, j) =arc, ordered pair $\neq (j, i)$
$i, j \in \mathcal{N}$

Graph and Flow



(i, j) outgoing arc for i, incoming arc for j

Complete graph = Graph with all possible arcs $\forall$ ordered pair of nodes

Assumptions :

(1) only one arc between a pair of ordered node

(2) only discreted graph

Path

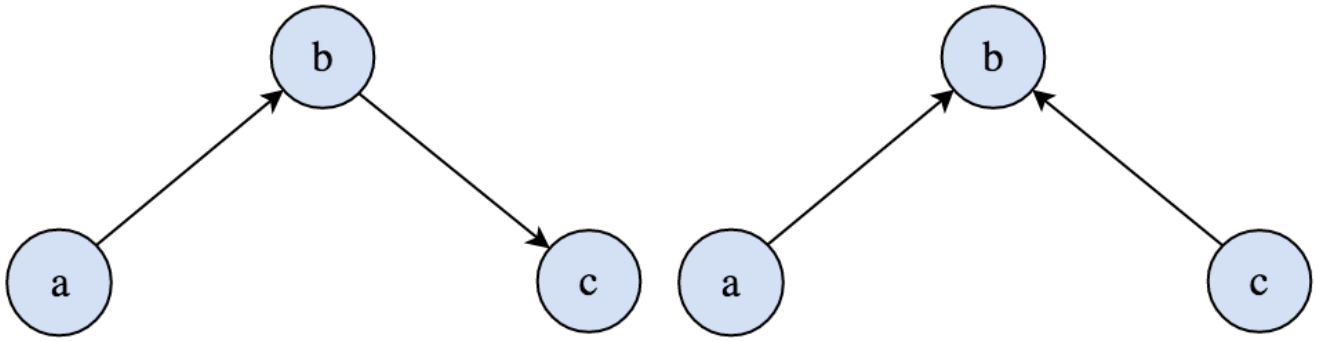
Path is a sequence of nodes $P = (m_1, m_2, ..., m_k), k \geqslant 2$
where $m_1$is the start node,$m_2$is the end node
with $(k-1)$ arc,

i th arc can be $\begin{cases} \overrightarrow{(m_i, m_{i+1})} & \text{forward arc} \\ \overleftarrow{(m_{i+1}, m_i)} & \text{backward arc} \end{cases}$

P can contain both forward and backward arcs
$P^+ =$ set of forward arcs of P
$P^- =$ set of backward arcs of P

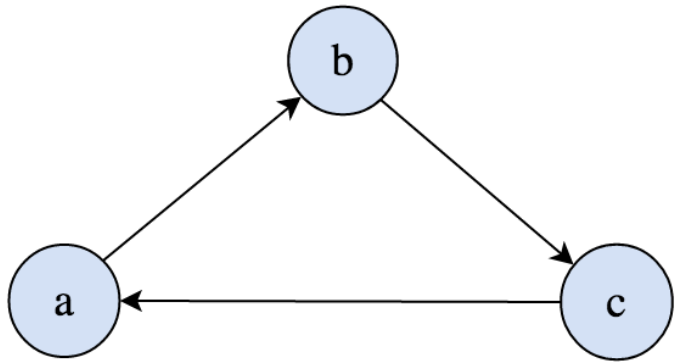
cycle = path P with same start and end nodes

Path



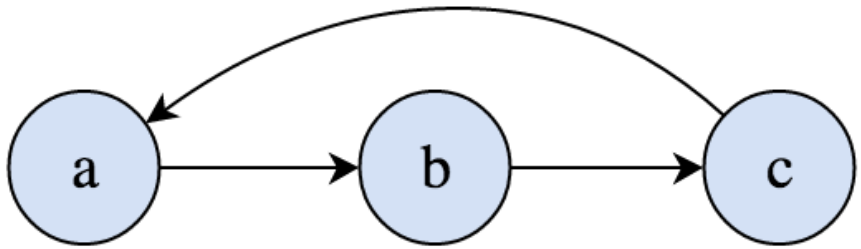
simple path = contains no repeated arcs and nodes except for the start and end nodes

simple cycle = simple path with same start and end nodes



Path

Hamiltonian cycle = simple cycle that contains all the nodes of the graph



acyclic graph= graph with no simple cycle

connected graph= $\forall i, j$ there is a path starting at i and end at j

strong connected graph= $\forall i, j$ there is a **forward path** starting at i and end at j

tree= connected acyclic graph

spanning tree = subgraph of $\mathcal{G}$ which is a tree and includes all the nodes of $\mathcal{G}$

Flow and Divergence

Flow of an arc (i,j) $x_{i,j} \in \mathbb{R}$, can be both $+$ and $-$

if $x_{i,j} < 0$, the flow moves in oppsite direction

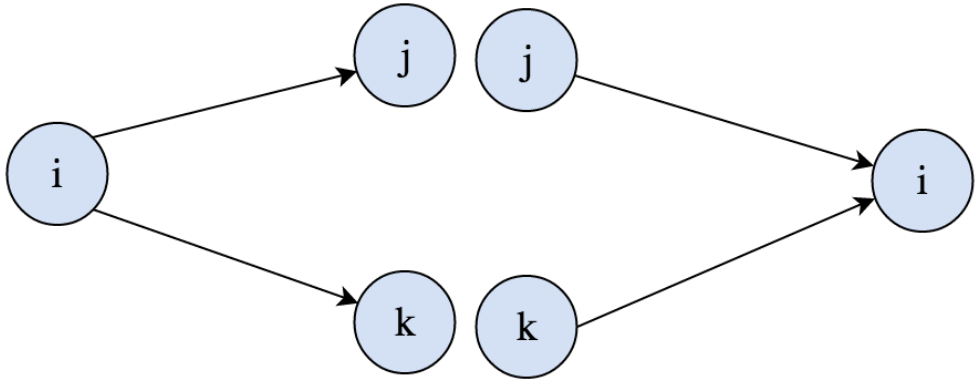
$x_{i,j}$ can change sign by changing arc direction
$\Rightarrow$ we can assume all the flows to be positive (if not, just change the arc sign)

flow vector is a set of flows $\{x_{i,j} | (i,j) \in \mathcal{A}\}$

divergence vector $y = [y_1, \ldots, y_N]$

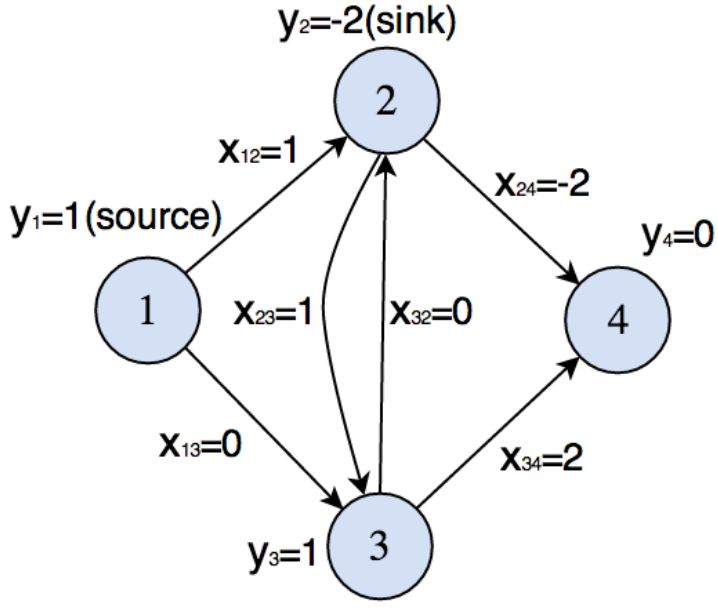
$$y_i \triangleq \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{i,j} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{j,i} \qquad \forall i \in \mathcal{N}$$

Flow and Divergence



departing flow from i arriving flow at i

source node $i, y_i > 0$
sink node $i, y_i < 0$
circulation
node $i, y_i = 0$ $\sum_{i \in \mathcal{N}} y_i = 0$



Flow and Divergence

It is often assumed that

$$b_{ij} \leqslant x_{ij} \leqslant c_{ij} \quad k_{(i,j)} \in \mathcal{A} \quad \bigstar$$

A path P let flow retrun x satisfying $\bigstar$ is unblocked

$$x_{ij} < c_{ij} \quad \forall (ij) \in P^{+}$$

$$x_{ij} > b_{ij} \quad \forall (ij) \in P^{-}$$

Network Flow Models: Examples

The Minimum Cost Flow Problem

Find a set of arc flows that minimize a linear cost function, subject to the constraints that they produce a given divergence vector and like within some given bounds

$$\min_x \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij}$$

$$\sum_{\{j|(i,j)\in\mathcal{A}\}} x_{i,j} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{j,i} = s_i, \quad \forall i \in \mathcal{N} \text{ conservation flow constraints}$$

$$b_{ij} \leqslant x_{ij} \leqslant c_{ij}, \qquad \forall (i,j) \in \mathcal{A}$$

capacity constraints a_{ij}, b_{ij}, c_{ij} *and* s_i are vector scalars

a_{ij} =cost of arc (ij)

b_{ij}, c_{ij} = flow bounds of (i,j)

$[b_{ij}, c_{ij}]$ = feasible flow range of (i,j)

s_i =supply of node i

Network Flow Models: Examples

The Minimum Cost Flow Problem

- A flow vector x satisfying both constraints is called **feasible**
- arcs can be thought as transportation links, they cost a_{ij} per unit transported
- Goal: move the product from source to distination at minimum cost

Network Flow Models: Examples

Example 1: The shortest path problem

Suppose the cost of a forward path to be the sum of the costs of its arcs. What is the forward path with the minimum cost ?

$$\min \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{i,j} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{j,i} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$$

$0 \leqslant x_{ij} \quad \forall (i,j) \in \mathcal{A} \quad s = \text{source node} \quad t = \text{distination node}$

Network Flow Models: Examples

Example 1: The Shortest Path Problem

Let us associate with any forward path P from s to t

The flow vector x

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

If x is feasible, then the cost of x $\sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij}$ is the length of P

It can be proved that if there is an optimal solution then the arc flows x_{ij} are either 0 or 1

Network Flow Models: Examples

Example 2: The Assignment Problem

How to match n persons to n objects, which only one person to an object is allowed, maximizing the benefit

Example 3: The Max-Flow Problem

Given a source s and a sink t, find the flow vector that makes the divergence of all nodes other than s and t equal to 0 while maximizing the divergence of s

Example: Maximizing the throughput of a transportation network on communication network

Example 4: The Transportation Problem

Similar to assignment problem of Example 2, except that supplies can be whatever and the number of sources and sinks need not to be equal

Network Flow Model with Convex Lost

The cost is convex

$$\min \sum_{(i,j)\in\mathcal{A}} f_{ij}(x_{ij})$$

Multicommodity Flow Problems

There are several flow types, which simultaneously use the network and are coupled through either the arc flow bounds, or through the cost function

Discrete Network Optimization Problems

The linear and convex problems can have addtional "interger constraints" on the arc flows, x_{ij} may not be just 0,1. When this happen, the computation of the solution becoms dynamic

Network Flow Models: Examples

1) Primal cost improvement

sequence of feasible flows is built to meet the optimum

2) Dual cost improvement

a) a dual problem is introduced, its decision variables are called prices

b) sequences of prices are built by a property called complementary stackness

c) the arc flows are connected by the prices

3) Auction

Sequences of prices are generated in a way that is reminiscent of real-life auctions. It can be seem as an approximate dual cost improvement process. However, also the flows are updated.
approximate form property: $\epsilon -$ *complementary stackness*

Definition

simple path flow= flow vector x that corresponds to sending a positive flow along a path P:

$$x_{ij} = \begin{cases} a & \text{if } (i,j) \in P^+ \\ -a & \text{if } (i,j) \in P^- \\ 0 & \text{otherwise} \end{cases} \qquad *$$

If P is a cycle, then we have a simple cycle flow

simple cycle flow x^s conforms to a flow vector x if the path P corresponding to x^s in$*$ is such that

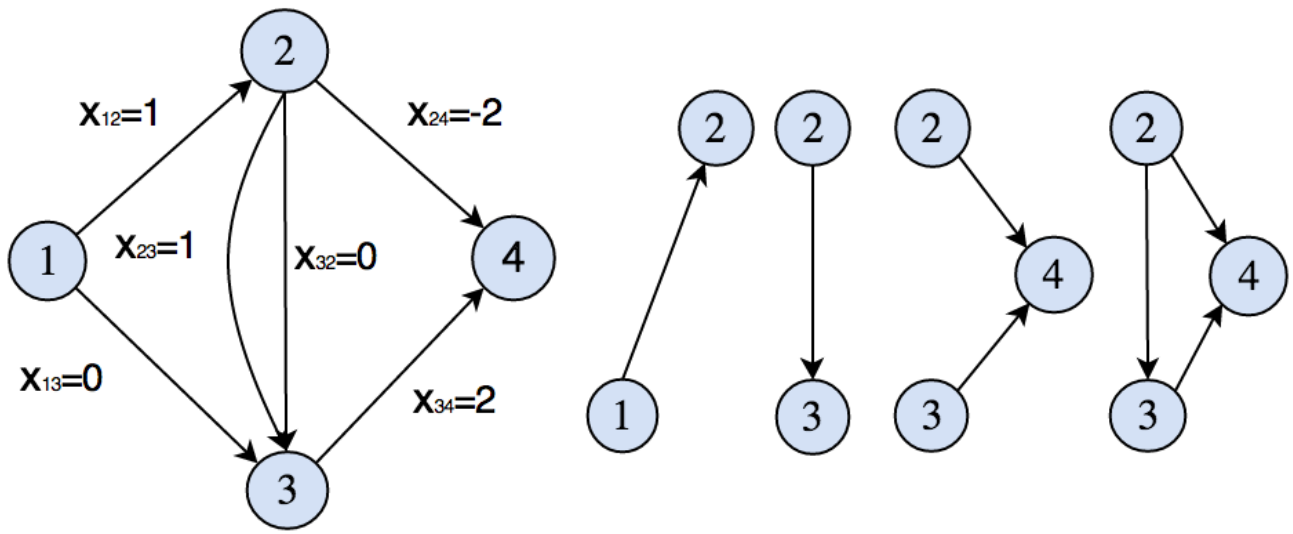
$$x_{ij} > 0 \quad \forall (i,j) \textit{ with } x^s_{i,j} > 0$$

$$x_{ij} < 0 \quad \forall (i,j) \textit{ with } x^s_{i,j} < 0$$

Proposition 1.1: (Conformal Realization Theorem)

A non-zero flow vector x can be decomposed into the sum of t simple path flow vectors $x^1, x^2, \ldots, x^t$ that conforms to x

Example



Moreover, if x is a circulation, the $x^1, x^2, \ldots, x^t$ can be chosen as simple cycle flows

Primal Cost Improved

Proposition 1.2

Consider the minimum cost flow problem. A flow vector x^* is optimal if and only if x^* is feasible and every simple cycle c that is unblocked with respect to x^* has nonnegtive cost:

$$\sum_{(i,j)\in\mathcal{C}^+} a_{ij} - \sum_{(i,j)\in\mathcal{C}^-} a_{ij} \geqslant 0$$

If x is feasible but not optimal and $\bar{x} \neq x$ is feasible and has smaller cost than x, then that one of the cycles in $\mathcal{G}$ conforming to $x - \bar{x}$ that has negative cost

Primal Cost Improved

Proposition 1.2

Let x and $\bar{x}$ be two feasible flow vector, $x \neq \bar{x}$
$Z \triangleq \bar{x} - x$ is a circulation

$$yi = \sum_{j(i,j)\in\mathcal{A}} Z_{ij} - \sum_{j(j,i)\in\mathcal{A}} Z_{ji} = s_j - s_i = 0$$

Suppose $\bar{x}$ has smaller cost than $x \Rightarrow$

$$\sum_{(i,j)\in\mathcal{A}} a_{ij} Z_{ji} < 0 \quad *$$

We can decompose Z as

$$Z = \sum_{x=1}^{K} w^k \xi^k \quad (ConformalRealizationTheorem)$$

$w^k \Rightarrow$ positive scalars $\xi^k \Rightarrow$ simple cycle flows on ξ with

Primal Cost Improved Cont'd

Proposition 1.2

$$\xi_{ij} = \begin{cases} 1 & Z_{(ij)} > 0 \\ -1 & Z_{(ij)} < 0 \end{cases}$$

Then

$$\sum_{(i,j)\in\mathcal{A}} a_{ij} Z_{ij} = \sum_{(i,j)\in\mathcal{A}} a_{ij} \sum_{k=1}^{K} w^k \xi_{ij}^x = \sum_{k-1}^{K} w^k \sum_{(i,j)\in\mathcal{A}} a_{ij} \xi_{ij}^x$$

$= \sum_{k=1}^{K} w^x c^k < 0 \quad for\ *$
where c^k is the cost of the simple cycle flow ξ^k
$\Rightarrow \quad K: \quad c^k < 0$ and a simple cycle ξ_k with path C_k

$$\exists c^x = \sum_{i,j\in\mathcal{A}} a_{ij} \xi_{ij}^x = \sum_{(i,j)\in\mathcal{C}_x^+} a_{ij} - \sum_{(i,j)\in\mathcal{C}_x^-} a_{ij}$$

Proof of Proposition 1.2

Let x^* be optimal flow vector and C be a simple cycle that is unblocked with respect to x^*. Then $\exists\ \epsilon > 0$, so that increasing the flow of arcs C^+ by ϵ and decreasing the arcs of C^- by ϵ , we have a feasible flow ξ which has cost equal to the cost of x^* plus ϵ times the cost of C

$$-[\sum_{(i,j)\in C^+} a_{ij}(x^*_{ij}+\epsilon) - \sum_{(i,j)\in C^-} a_{ij}\bar{x_{ij}}] - Rest$$

$$+[\sum_{(i,j)\in C^+} a_{ij}(x^*_{ij}+\epsilon) - \sum_{(i,j)\in C^-} a_{ij}(\bar{x_{ij}}+\epsilon)] + Rest$$

$$= \sum_{(i,j)\in C^+} a_{ij} - \sum_{(i,j)\in C^-} a_{ij} > 0$$