1 Introduction

- Network optimization Problems (NOP) arises naturally in analysis and design of large systems
- How to select routes that minimize the cost of transfer from supply to demand
- There are four main classes of NOP
  1) Transshipment or minimum cost flow problem
     single commodity and linear cost function
     shortest path, max flow, assignment, transportation
  2) Single commodity network flow problem with convex cost
  3) Multy commodity NOP with linear or convex cost
     generalize the previous 1) +2)
  4) Discrete network optimization
     quantities transfered over the network take one or many discrete values

the focus of the course is on modeling + analysis + solution methods + examples

2 Graphs and Flow

$G = (N, A)$ directed graph

N: set of nodes

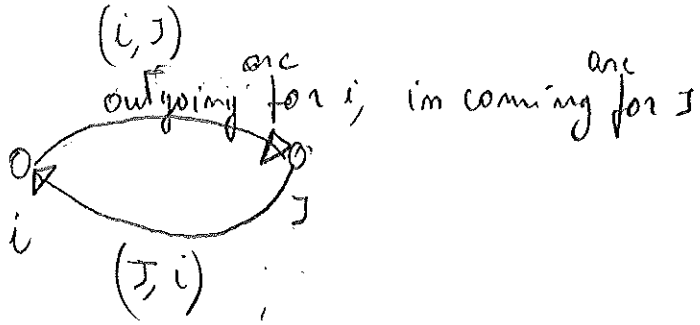
A: set of pairs of distinct nodes of N called arcs

$|N| = N$ number of nodes

$|A| = A$ number of

(i,j) = arc, ordered pair $\neq (j,i)$

$i,j \in N$



Complete graph = graph with all possible arcs $\forall$ ordered pair of nodes

Assumptions : 1) only one arc between a pair of ordered nodes

2) only directed graph

Path

is a sequence of nodes $P = (n_1, n_2, \dots, n_K)$ $K \geq 2$

start node: n_1 ; end node: n_K

with $K-1$ arc

i-th arc can be $\begin{cases} (n_i, n_{i+1}) & \text{forward arc} \\ (n_{i+1}, n_i) & \text{backward arc} \end{cases}$

P can contain both forward and backward arcs

P^+ = set of forward arcs of P

P^- = set of backward arcs of P

cycle = path p/ with same start and end nodes

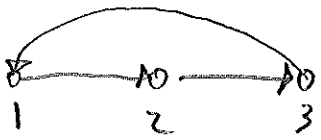


simple path = contains no repeated arcs and nodes except for the start and end nodes

simple cycle = simple path with same start and end nodes



Hamiltonian cycle = simple cycle with all forward arcs containing all nodes of the graph



acyclic graph = graph with no simple cycle

connected graph = $\forall i, j$, $\exists$ p starting at i and ending at j

strongly connected graph = $\forall i, j$, $\exists$ forward path

[$\forall$ connected graph with nodes having even degree, then is a cycle containing all the arcs only once
Euler graph]

tree = connected acyclic graph

spanning tree = sub graph of G that is a tree and includes all the nodes of G

Flow of an arc (i,j) $x_{ij} \in \mathbb{R}$ can be both + and −

if $x_{ij} < 0$, the flow moves in opposite direction

x_{ij} can change sign by changing arc direction

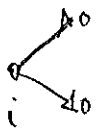
$\Rightarrow$ we can assume all the flows to be positive (if not, just change arc sign)

flow vector set of flows $\{x_{ij} \mid (i,j) \in A\}$

divergence vector $y = [y_1, \dots y_N]$

$$y_i \triangleq \sum_{\{j \mid (i,j) \in A\}} x_{ij} - \sum_{\{j \mid (j,i) \in A\}} x_{ji} \quad \forall i \in N$$

diverging flow from i



converging flow at i

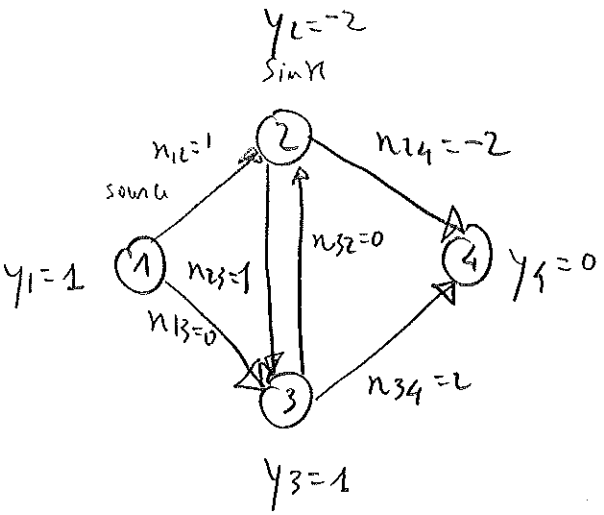


source node i $y_i > 0$

sink node i $y_i < 0$

circulation node i $y_i = 0$

$$\sum_{i \in N} y_i = 0$$



It is often assumed that

$$b_{ij} \le x_{ij} \le c_{ij} \quad \forall (i,j) \in A \qquad (*)$$

A path P having flow vector x satisfying $(*)$ is unblocked

$$x_{ij} < c_{ij} \quad \forall (i,j) \in P^+$$

$$x_{ij} > b_{ij} \quad \forall (i,j) \in P^-$$

Network flow Models : Examples

Find a set of arc flows minimizing a cost function subject to constraints that produce a given divergence and lie within some bound — The Minimum Cost Flow Problem

$$\min_{x} \sum_{(i,j)\in A} a_{ij} x_{ij}$$

$$\sum_{\{j \mid (i,j)\in A\}} x_{ij} - \sum_{\{j \mid (j,i)\in A\}} x_{ji} = s_i \quad \forall i \in N$$

→ conservation flow constraint

$$b_{ij} \le x_{ij} \le c_{ij} \quad \forall (i,j) \in A$$

→ capacity constraint

a_{ij}, b_{ij}, c_{ij} and s_i are known scalars

a_{ij} = cost of arc (i,j)

b_{ij}, c_{ij} = flow bounds of (i,j)

$[b_{ij}, c_{ij}]$ = feasible flow range of (i,j)

s_i = supply of node i

- A flow vector x satisfying both constraints is called feasible
- arcs can be thought as transportation links that cost a_{ij} per unit of transported good
- Goal: move the product from source to destination at minimum cost

Example 1: the shortest path problem

Suppose the cost of a forward path be the sum of the costs of its arcs. What is the forward path having the minimum cost?

Example a_{ij} = delay in communication

$$\min \sum_{(i,j)\in A} a_{ij} x_{ij}$$

$$\text{s.t.} \sum_{\{j \mid (i,j)\in A\}} x_{ij} - \sum_{\{j \mid (j,i)\in A\}} x_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$$

$$0 \le x_{ij} \quad \forall (i,j) \in A$$

s = source node

t = destination node

Let us associate with any forward path P from s to t the flow vector x:

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

If x is feasible, then the cost of x $\left(\sum_{(i,j)\in A} a_{ij} x_{ij}\right)$ is the length of P

It can be proved that if there is an opt sol, then the arc flows x_{ij} are either 0 or 1

Example 2 : the assignment problem

How to match n persons to n objects, where only one person to an object is allowed, maximizing the benefit

Example 3 : the max flow problem

Given a source s and a sink t, find the flow vector that makes the divergence of all nodes other than s and t equal to 0 while maximizing the divergence of s

Exemple. maximizing the throughput of a Transportation network on computer network

Example 4 : the transportation problem

Similar to assignment problem of Example 2, except that supplies can be whatever and the number of sources and sinks need not to be equal

5 Network Flow Model with Convex cost

the cost is convex

$$\min \sum_{(i,j) \in A} f_{ij}(x_{ij})$$

6 Multicommodity Flow Model

there are several type of flows $x_{ij}(m)$

7 Discrete Network Optimization Problems

the linear and convex problems mentioned can have additional "integrality" constraints on the arc flows, x_{ij} and not be just 0, 1. When this happen, the computation of the solution becomes dramatic.

8 Network Flow Algorithms

1) Primal cost improvement
   - sequence of feasible flows are built to reach the optimum

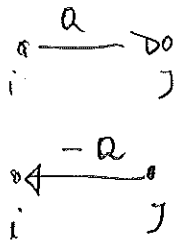
2) Dual cost improvement
   a) a dual problem is introduced. Its decision variables are called prices
   b) sequences of prices are built by a property called complementary slackness
   c) the arc flows are recovered by the prices

3) Auction
   Sequence of prices are generated in a manner that resembles an auction. It can be seen as an approximate dual cost improvement process. However, also the flows are updated.
   Key property: ϵ - complementary slackness

Definition

Simple path flow = flow vector x that corresponds to sending a positive flow along a path P:

$$x_{ij} = \begin{cases} a & \text{if } (i,j) \in P^+ \\ -a & \text{if } (i,j) \in P^- \\ 0 & \text{otherwise} \end{cases} \quad (*)$$



If P is a cycle, then we have a simple cycle flow

Simple path flow x^s conforms to a flow vector x if the path P corresponding for x^s in $(*)$ is such that

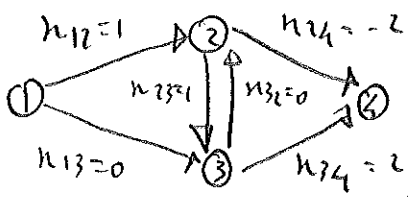
$x_{ij} > 0 \quad \forall (i,j)$ with $x^s_{ij} > 0$

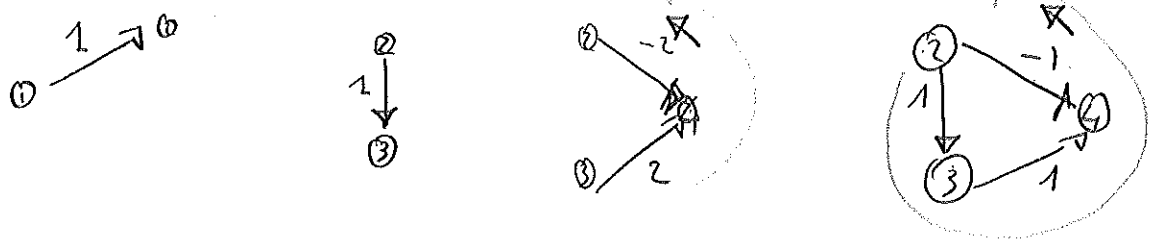
$x_{ij} < 0 \quad \forall (i,j)$ with $x^s_{ij} < 0$

Proposition (Conformal Realization theorem)

A non zero flow vector x can be decomposed into the sum of t simple path flow vectors $x^1, x^2, \dots x^t$ that conform to x

Example





Moreover, if x is a circulation, then
$x^1, x^2, \dots, x^t$ can be chosen as simple cycle flows

9) Primal Cost Improvement

Proposition 1.2 Consider the minimum cost flow problem. A flow vector x^* is optimal if and only if x^* is feasible and every simple cycle C that is unblocked w.r.t. x^* has non negative cost:

$$\sum_{(i,j)\in C^+} a_{ij} - \sum_{(i,j)\in C^-} a_{ij} \geq 0$$

Lemma If x is feasible but not optimal and $\bar{x} \neq x$ is feasible and has smaller cost than x, then there is a simple cycle in $\mathcal{G}$ conforming to $x - \bar{x}$ that has negative cost

Let x and $\bar{x}$ be two feasible flow vector $x \neq \bar{x}$

$z \triangleq \bar{x} - x$ is a circulation

$$y_i = \sum_{j|(i,j)\in A} z_{ij} - \sum_{j|(j,i)\in A} z_{ji} = s_i - s_i = 0$$

Suppose $\bar{x}$ has smaller cost than x $\Rightarrow$

$$\sum_{(i,j)\in A} a_{ij} z_{ji} < 0 \qquad (*)$$

We can decompose z as

$$z = \sum_{k=1}^{K} w^k \xi^k$$ (Conformal realization theorem)

w^k: positive scalars; ξ^k: simple cycle flows on $\mathcal{G}$ with

$$\xi^k_{ij} = \begin{cases} 1 & z_{ij} > 0 \\ -1 & z_{ij} < 0 \end{cases}$$

then

$$\sum_{(i,j)\in A} a_{ij} z_{ij} = \sum_{(i,j)\in A} a_{ij} \sum_{k=1}^{K} w^k \xi^k_{ij} = \sum_{k=1}^{K} w^k \sum_{(i,j)\in A} a_{ij} \xi^k_{ij}$$

$$= \sum_{k=1}^{K} w^k c^k < 0 \quad \text{for } \circledast$$

$\uparrow$ cost of the simple cycle flow φ^k

$\Rightarrow \exists\, k : c^k < 0$ and a simple cycle φ^k with path G^k

$$c^k = \sum_{(i,j) \in A} a_{ij} \varphi_{ij}^k = \sum_{(i,j) \in G_k^+} a_{ij} - \sum_{(i,j) \in G_k^-} a_{ij}$$

Proof of Prop 1.2

Let x^* be optimal and G a simple cycle that is unblocked w.r.t. x^* then $\exists\, \varepsilon > 0$ so that increasing the flow of arcs G^+ by ε and decreasing by ε the arcs of G^-, we have a feasible flow $\bar{x}$ whose costs of the flow of x^* and $\bar{x}$ over G are

$$-\left[\sum_{(i,j) \in G^+} a_{ij} x_{ij}^* - \sum_{(i,j) \in G^-} a_{ij} \bar{x}_{ij}\right] - \text{Rest}$$

cost of the optimization problem

$$+\left[\sum_{(i,j) \in G^+} a_{ij}(\bar{x}_{ij} + \varepsilon) - \sum_{(i,j) \in G^-} a_{ij}(\bar{x}_{ij} + \varepsilon)\right] + \text{Rest} > 0$$

$$= \sum_{(i,j) \in G^+} a_{ij} \varepsilon + \sum_{(i,j) \in G^-} a_{ij} \varepsilon > 0$$