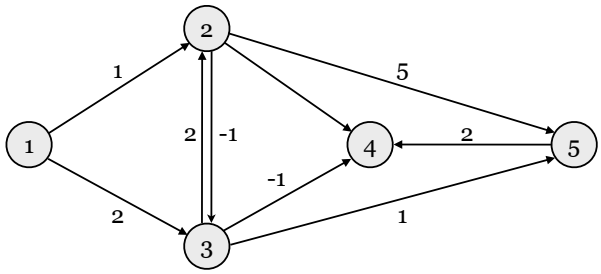
Homework: Introduction

Note: Every homework has 10 credit points. To pass the homework, you need to reach 50% of the points. In this homework, only the first four exercises have credit points.

1. Consider the graph and the flow vector of the following figure:



(a) Enumerate the simple paths and the simple forward paths that start at node 1. (0.5p)

(b) Enumerate the simple cycles and the simple forward cycles of the graph. (0.5p)

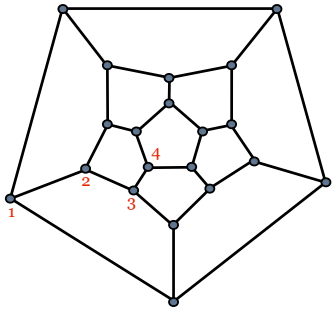
(c) Is the graph strongly connected? (0.5p)

(d) Show that the divergences of the nodes add to zero. (0.5p)

(e) Give an example of a simple path flow that starts at node 1, ends at node 5, involves 4 arcs, and conforms to the given flow vector. (0.5p)

2. **(a)** What is a Hamiltonian cycle? (0.5p)

(b) Find a Hamiltonian cycle on the graph below, using as first four nodes the ones marked 1-4 (all arcs are considered bidirectional). (1.5p)



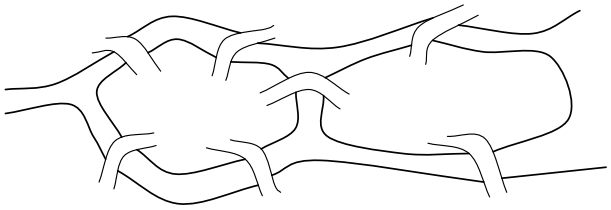
(c) What is an acyclic graph? (0.5p)

3. **(a)** What is an Euler cycle (or Eulerian cycle or Euler tour)? (0.5p)

(b) What is an Euler path (or Euler walk)? (0.5p)

(c) The first paper on graph theory was written by Leonhard Euler in 1736, dealing with the bridges of Königsberg. The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River. It included two large islands

which were connected to each other and the mainland by seven bridges (shown schematically below).



The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge was to be crossed completely every time. One could not walk half way onto the bridge and then turn around and later cross the other half from the other side.
Why does the "Seven Bridges of Königsberg" does not have a solution? (1.5p)

4. Consider the following variant of the minimum cost flow problem

$$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\ \text{subject to} \quad & \underline{s}_i \le \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{j|(j,i)\in\mathcal{A}} x_{ji} \le \bar{s}_i, \qquad \forall i \in \mathcal{N} \\ & b_{ij} \le x_{ij} \le c_{ij}, \qquad \forall (i,j) \in \mathcal{A}, \end{aligned}$$

where the bounds $\underline{s}_i$ and $\bar{s}_i$ on the divergence of node i are given. Show that the problem can be converted to a standard (equality constrained) minimum cost flow problem by adding an extra node B and an arc (B,i) from this node to every other node i, with feasible flow range $[0, \bar{s}_i - \underline{s}_i]$. (2.5p)

5. A dating website receives data from n men and m women. These data determine what pairs of men and women are mutually compatible. Since the website's commission is proportional to the number of dates it arranges, it would like to find the maximum number of compatible couples that can be formed. Formulate this problem as an assignment problem.

6. The bubble sort algorithm maintains a an ordered set of the numbers $\{a_1, a_2, \ldots, a_n\}$ that it rearranges through a sequence of several passes over the set. In each pass, the algorithm examines every pair of elements (a_k, a_{k+1}) for each $k = 1, 2, \ldots, (n-1)$, and if the pair is out of order (i.e., $a_k > a_{k+1}$), it swaps the positions of this elements. The algorithm terminates when it makes no swap during one entire pass.

(a) Show that the bubble sort performs at most n passes and runs in $O(n^2)$ algorithm.

(b) Show that the bubble sort is a $\Theta(n^2)$ algorithm.